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Digital lines with irrational slopes

Hanna Uscka-Wehlou

Abstract

How to construct a digitization of a straight line and be able to recognize a straight line in a set of pixels are very important topics in computer graphics. The aim of the present paper is to give a mathematically exact and consistent description of digital straight lines according to Rosenfeld's definition. The digitizations of the lines with slopes $0 < a < 1$, where a is irrational, are considered. We formulate a definition of digitization runs, formulate and prove theorems containing necessary and sufficient conditions for digital straightness. The proof was successfully constructed using only methods of elementary mathematics. The developed and proved theory can be used in the research into the theory of digital lines, their symmetries, translations etc.

Keywords: Digital geometry; Theory of digital lines; Irrational slope; Continued fractions

1. Introduction

Our aim here is to give a mathematically exact and consistent description of digital straight lines according to Rosenfeld's definition (1974). We will consider the digitizations of the lines with slopes $0 < a < 1$ where a is irrational. The theory for such lines appears to be very elegant and simple. When treating rational slopes together with irrational, however, we are forced to deal with special cases and exceptions which would make the theory less clear.

A detailed review on digital straightness can be found in Rosenfeld & Klette (2001). Necessary and sufficient conditions for digital straightness are formulated there; see for example Wu's theorem from 1982 (Theorem 3.5 in Rosenfeld & Klette (2001)). Different approaches and kinds of proofs (algorithms, using word theory, etc.) are also discussed there.

There has been done a lot of research concerning digital straightness lately; see for example Reveillès (1991), Debled (1995), Vittone (1999). They describe digital lines with rational slopes. Lines with irrational slopes, however, have not got enough attention in scientific papers. There are very few researchers dealing with this subject. Some of them have used the link between combinatorics on words and digital lines and planes; see Arnoux et al. (2004), Jamet (2004). We present a description of digital lines with irrational slopes without using any advanced theories.

Stephenson and Litow (2000, 2001) have described fast algorithms for drawing digital lines with rational slopes. Although the present paper covers the theory for lines with irrational slopes, one can easily use it as a basis for the formal proof of the results for lines with rational slopes presented by them.

The central role in the construction of the theory presented here is played by Lemma 3.6. The most important definitions are Definition 3.4 and Definition 3.7. The main results are formulated in Theorem 3.13 and Corollary 3.14 (necessary conditions to be a digital line with irrational slope). The corollary is more practically useful than the theorem itself. A sufficient condition to be a digital line with irrational slope is formulated in Theorem 3.17.

The proof of the necessary condition for digital straightness is based on as elementary mathematics as possible, without resorting to algorithms.

2. Rosenfeld's digitization

Rosenfeld's definition of the digitization of a straight line can be presented as follows. See also Rosenfeld (1974) and Melin (2003).

Rosenfeld's plane can be identified with $\mathbf{Z}^2$. With each point (k, n) of this plane we can associate the following two subsets of $\mathbf{R}^2$:

$$S_R(k,n) = \left]k - \tfrac{1}{2}, k + \tfrac{1}{2}\right] \times \left]n - \tfrac{1}{2}, n + \tfrac{1}{2}\right]$$

and

$$C_R(k,n) = \left(\{k\} \times \left]n - \tfrac{1}{2}, n + \tfrac{1}{2}\right]\right) \cup \left(\left]k - \tfrac{1}{2}, k + \tfrac{1}{2}\right] \times \{n\}\right).$$

We will call them *R-squares* and *R-crosses* in (k, n) respectively. One can easily see that the R-squares form a partition of $\mathbf{R}^2$, i.e.:

$$\mathbf{R}^2 = \bigcup_{(k,n)\in\mathbf{Z}^2} S_R(k,n), \quad \text{and}$$

$$(k_1, n_1) \neq (k_2, n_2) \;\Rightarrow\; S_R(k_1, n_1) \cap S_R(k_2, n_2) = \emptyset.$$

Rosenfeld's digitization of a straight line l (which we will denote by $D_R(l)$) is the set of all (k, n) in $\mathbf{Z}^2$ for which the intersection of l and $C_R(k, n)$ is not empty:

$$D_R(l) = \{(k,n) \in \mathbf{Z}^2;\ \ l \cap C_R(k,n) \neq \emptyset\}.$$

For some lines, such as $y = x + \frac{1}{2}$, we obtain thick digitizations which can be adjusted to one pixel thin lines (*naive lines* according to Reveillès (1991)) by elimination of some pixels; see Melin (2003: section 1) and Kiselman (2004: Theorem 6.1).

We will discuss the digitization of the positive half line only, i.e., the digitization of $y = ax$ where $x > 0$ (*rays* in Rosenfeld & Klette (2001)), since the digitization of the negative half line can be derived by symmetries.

It is worth mentioning that the slope is the most important feature characterizing a digital line:

- Two lines $y = a_1x + b_1$ and $y = a_2x + b_2$ where $a_1 \neq a_2$ cannot have the same digitization, because $|a_1x + b_1 - (a_2x + b_2)| \to \infty$ when $x \to \infty$. The slope is thus determined by the digitization and this is why we can say that a digital line has a slope.

- Two lines $y = ax + b_1$ and $y = ax + b_2$ where $b_1 \neq b_2$ can have the same digitization, like for example lines $y = \frac{2}{5}x$ and $y = \frac{2}{5}x + \frac{1}{40}$. Parallel translated lines cannot always be distinguished in their digitized form.

Exact description of those two items can be found in Rosenfeld & Klette (2001), formulated in Theorem 1.2 (theorem of Bruckstein).

3. The necessary and sufficient conditions

We are mainly interested in straight lines with an irrational slope between 0 and 1 which pass through the origin, i.e., lines $y = ax$ where $0 < a < 1$ and a is irrational. Digitizations of lines with irrational slopes $a < 0$ and $a > 1$ can be obtained by a change of coordinates; see Rosenfeld (1974).

In order to make it easier to handle descriptions and equations, we will modify the definition of the R-digitization by changing the definitions of R-squares and R-crosses in the following way:

$$S_{R'}(k, n) = \left]k - \tfrac{1}{2}, k + \tfrac{1}{2}\right] \times]n - 1, n] = S_R\left(k, n - \tfrac{1}{2}\right)$$

and

$$C_{R'}(k, n) = (\{k\} \times]n - 1, n]) \cup \left(\left]k - \tfrac{1}{2}, k + \tfrac{1}{2}\right] \times \{n - \tfrac{1}{2}\}\right) = C_R\left(k, n - \tfrac{1}{2}\right).$$

We call these *R′-squares* and *R′-crosses* respectively. Then we define the R′-digitization of the line l as follows:

$$D_{R'}(l) = \{(k, n) \in \mathbf{Z}^2;\ \ l \cap C_{R'}(k, n) \neq \emptyset\} = \{(k, \lceil ak \rceil);\ \ k \in \mathbf{Z}\}.$$

Fig. 1 shows a comparison of the two digitizations.
The R′-digitization of the line with equation $y = ax$ is equal to the R-digitization of $y = ax + \frac{1}{2}$:

$$(k, n) \in D_R(y = ax + \tfrac{1}{2}) \ \Leftrightarrow\ n - \tfrac{1}{2} < ak + \tfrac{1}{2} \leqslant n + \tfrac{1}{2} \ \Leftrightarrow\ n - 1 < ak \leqslant n$$
$$\Leftrightarrow\ (k, n) \in D_{R'}(y = ax).$$

This is also illustrated in Fig. 1.
If $0 < a < 1$, then $f(x) = ax$ is a function and it is increasing, so the R′-digitization of the line l with equation $y = ax$ consists of horizontal *runs*:

$$\mathrm{run}(n) = \{(k, n) \in D_{R'}(l)\} = \{(k, n) \in \mathbf{Z}^2;\ n - 1 < f(k) \leqslant n\}$$

(hence $\lceil f(k) \rceil = n$), where the second coordinate gives an enumeration of R′-digitization runs. We can also talk about the first, second,..., last element of a run, using the order in $\mathbf{Z}$ on the first coordinate. For example, the last element of $\mathrm{run}(0)$ is $(0, 0)$, the first element of $\mathrm{run}(1)$ is $(1, 1)$, since $a \in]0, 1[$ (see Fig. 2).

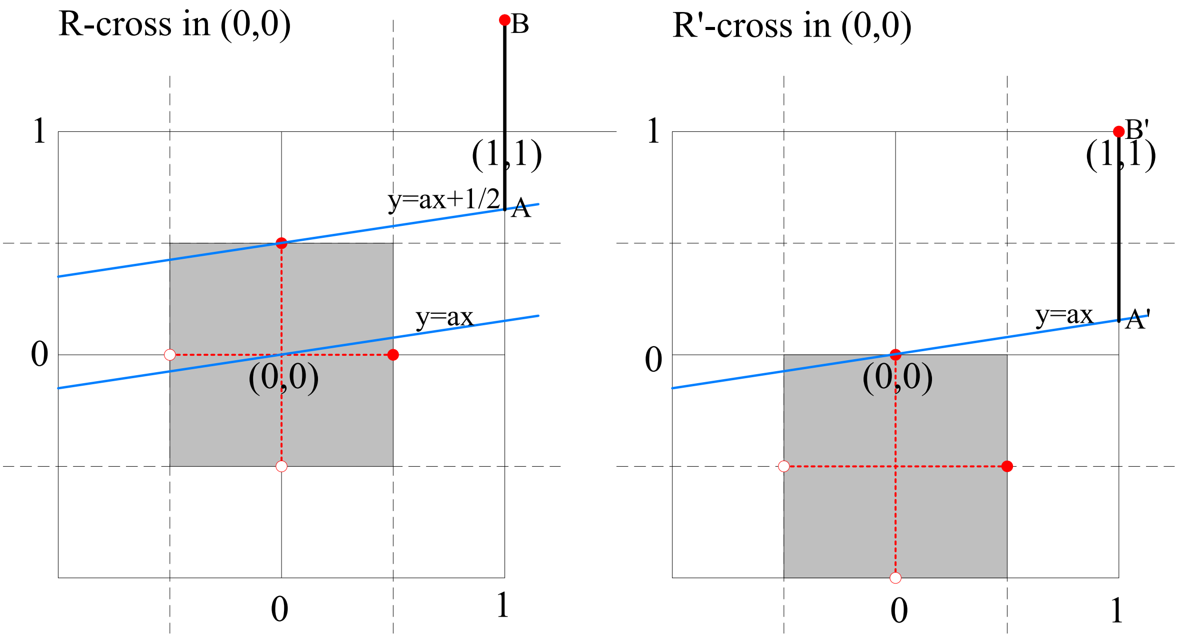
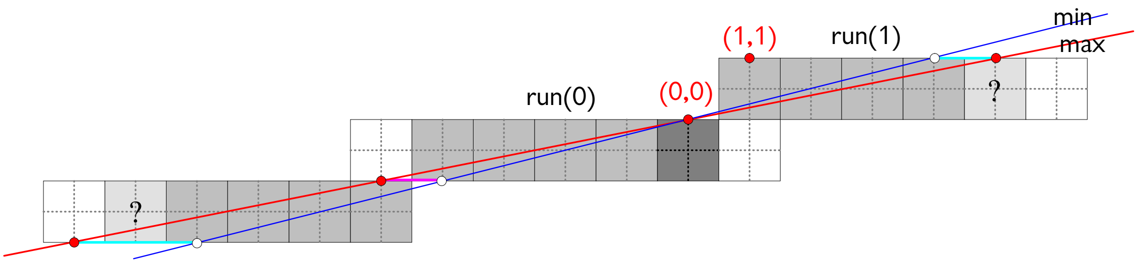


Figure 1. R-cross and R′-cross in $(0,0)$. $AB = A'B'$, so $D_{R'}(y = ax) = D_R(y = ax + \frac{1}{2})$.



Run(0) with length 5 belongs to the R'-digitizations of the lines y=ax lying between the two lines as on the picture.

The only possible lengths of run(1) in the digitizations containing run(0) defined above are 4 or 5.

Figure 2. Digitization runs

We define the *length of a run* as the number of its elements, thus its cardinality $\mathrm{card}(\mathrm{run}(n))$.

First we will describe the R′-digitization on the level of runs as defined above. From now on, when we write *digitization*, we refer to the R′-digitization. Because we only analyze straight lines $y = ax$ (where $0 < a < 1$, and a is irrational) for $x > 0$, we begin the description of the digitization with run(1). We use the notation $\mathbf{N}^+ = \mathbf{N} \setminus \{0\}$.

The following lemma is useful for further calculations:

Lemma 3.1. *If* $\sigma \neq 0$, *then for every number* $i \in \mathbf{N}^+$ *the value of* $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ *is one of two consecutive natural numbers* $\lfloor \frac{1}{\sigma} \rfloor$ *and* $\lfloor \frac{1}{\sigma} \rfloor + 1$.

We observe that $\lfloor \frac{i}{\sigma} \rfloor$ is increasing (or decreasing, if $\sigma < 0$) on average like $\frac{i}{\sigma}$ (i.e., we have $\lfloor \frac{i}{\sigma} \rfloor / \frac{i}{\sigma} \to 1$ when $i \to \infty$), thus the average of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ over intervals $[1,k]_{\mathbf{Z}}$ with $k \to \infty$ is $\frac{1}{\sigma}$, meaning

$$\lim_{k\to\infty} \tfrac{1}{k} \sum_{i=1}^{k} \left(\left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor \right) = \tfrac{1}{\sigma}.$$

Lemma 3.1 says that $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ for $i \in \mathbf{N}^+$ can have only one of the two possible values $\left\lfloor \frac{1}{\sigma} \right\rfloor$ and $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$. This means that $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ takes the value of $\left\lfloor \frac{1}{\sigma} \right\rfloor$ and $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$ with such frequencies that the average is $\frac{1}{\sigma}$. This implies that $\left\lfloor \frac{1}{\sigma} \right\rfloor$ must appear with frequency $1 - \operatorname{frac}\left(\frac{1}{\sigma}\right)$ and $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$ with frequency $\operatorname{frac}\left(\frac{1}{\sigma}\right)$, because

$$\left(1 - \operatorname{frac}\left(\tfrac{1}{\sigma}\right)\right) \left\lfloor \tfrac{1}{\sigma} \right\rfloor + \operatorname{frac}\left(\tfrac{1}{\sigma}\right) \left(\left\lfloor \tfrac{1}{\sigma} \right\rfloor + 1\right) = \tfrac{1}{\sigma}.$$

By frequency of value $\left\lfloor \frac{1}{\sigma} \right\rfloor$ (resp. $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$) we mean the limit (when $k \to \infty$) of the number of these $i \in [1,k]_{\mathbf{Z}}$ for which $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor$ (resp. $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$) divided by k. Expressed symbolically, the frequency of value $\left\lfloor \frac{1}{\sigma} \right\rfloor$ is

$$\lim_{k\to\infty} \tfrac{1}{k}\operatorname{card}(S(k)), \quad \text{where} \quad S(k) = \{i \in [1,k]_{\mathbf{Z}};\ \left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor = \left\lfloor \tfrac{1}{\sigma} \right\rfloor\}.$$

Later (in Lemma 3.6) we will indicate in detail for which i we get which values of $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$.

Proof. For each $x, y \in \mathbf{R}$ we have:

$$\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if} \quad \operatorname{frac}(x) + \operatorname{frac}(y) < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if} \quad \operatorname{frac}(x) + \operatorname{frac}(y) \geqslant 1. \end{cases}$$

Taking $x = \frac{i-1}{\sigma}$ and $y = \frac{1}{\sigma}$, we get the assertion of the lemma. □

We can use Lemma 3.1 for the proof of the following proposition about the digitization runs:

Proposition 3.2. *For the digitization of the half line $y = ax$ (where $x > 0$, a is irrational and $0 < a < 1$) we have:*

1. *The length of the* $\operatorname{run}(j)$ *for $j \in \mathbf{N}^+$ is equal to* $\left\lfloor \frac{j}{a} \right\rfloor - \left\lfloor \frac{j-1}{a} \right\rfloor$.
2. *There are exactly two run lengths in the digitization: $\left\lfloor \frac{1}{a} \right\rfloor$ (short runs) and $\left\lfloor \frac{1}{a} \right\rfloor + 1$ (long runs).*
3. *The first run is short.*

Proof. In this proof, i counts the elements within runs, j counts the runs. Let $j \in \mathbf{N}^+$ be given. We examine the function $f(x) = ax$ for all integer arguments greater than or equal to 1, which we will call i (thus $i \in \mathbf{N}^+$). According to the definition of the R′-digitization, we have:

$$(i,j) \in \operatorname{run}(j) \quad \Leftrightarrow \quad j - 1 < f(i) \leqslant j \quad \Leftrightarrow \quad \tfrac{j-1}{a} < i \leqslant \tfrac{j}{a} \quad \Leftrightarrow \quad \left\lfloor \tfrac{j-1}{a} \right\rfloor < i \leqslant \left\lfloor \tfrac{j}{a} \right\rfloor$$

(the second equivalence we get because $a > 0$, the third one because $i \in \mathbf{Z}$). This means that the $\operatorname{run}(j)$ for $j \in \mathbf{N}^+$ begins in $\left(\left\lfloor \frac{j-1}{a} \right\rfloor + 1, j\right)$ and ends in $\left(\left\lfloor \frac{j}{a} \right\rfloor, j\right)$, and this means that the length of $\mathrm{run}(j)$ for $j \in \mathbf{N}^+$ is equal to $\left\lfloor \frac{j}{a} \right\rfloor - \left\lfloor \frac{j-1}{a} \right\rfloor$, which proves assertion 1. In particular, for $j = 1$: the first run begins in $(1, 1)$ and ends in $\left(\left\lfloor \frac{1}{a} \right\rfloor, 1\right)$, so its length is $\left\lfloor \frac{1}{a} \right\rfloor$. This means that the first run is short for all a, which proves assertion 3. Assertion 2 of the proposition follows now from Lemma 3.1, by replacing σ with a. $\square$

Our aim in this paper is a full description of the digitization of a given straight half line l $(x > 0)$ with equation $y = ax$, where $0 < a < 1$ and a is irrational. The first level of digitization has already been discussed. The notion of digitization level k will be formulated later. The *digitization parameters*, which will be defined now, are sufficient to derive a complete description of the digitization of the line they come from. In the definition of the digitization parameters we will use the following *modification operation* $\cdot^\wedge : [0, 1] \to [0, \frac{1}{2}]$:

Definition 3.3. *For* $t \in [0, 1]$ *we define* $t^\wedge = \min(t, 1 - t)$.

Definition 3.4. *For* $y = ax$ *where* $0 < a < 1$ *and* a *is irrational, the* digitization parameters *are:*
$\sigma_1 = \mathrm{frac}\left(\frac{1}{a}\right)$,
$\sigma_k = \mathrm{frac}\left(1/\sigma_{k-1}^\wedge\right)$ *for all natural* $k > 1$.
For $j \in \mathbf{N}^+$, σ_j *and* $\sigma_j^\wedge$ *are the* digitization parameters *and* modified digitization parameters *of the* digitization level j *respectively.*

For an irrational slope a there exist parameters σ_j for all $j \in \mathbf{N}^+$. We have $0 < \sigma_j < 1$ and σ_j is irrational for all $j \in \mathbf{N}^+$. The definition of σ_1 differs from the definition of σ_j for natural $j \geqslant 2$, since digitization runs of the first level as described in Proposition 3.2 are built of elements of *one* kind (elements of $\mathbf{Z}^2$) while the runs on all the following digitization levels will be composed of *two* kinds of elements (short and long). We will use the digitization parameters to compute the length of the runs on all the levels. To compute it correctly, it is important to know which kind of elements is the most frequent on each level and how to use the digitization parameters in both cases, i.e., depending on whether the short element or the long element is the most frequently occurring. It is obvious that $0 < \sigma_k^\wedge < \frac{1}{2}$ for each $k \in \mathbf{N}^+$.

We introduce an auxiliary function which counts for each digitization level k where $k \in \mathbf{N}^+$ all the previous levels (i.e., levels with numbers $1 \leqslant i \leqslant k - 1$) with digitization parameters fulfilling the condition $\sigma_i < \frac{1}{2}$:

Definition 3.5. *For a given straight line* l *with equation* $y = ax$, *where* $0 < a < 1$ *and* a *is irrational, we define function* $\mathrm{Reg} : \mathbf{N}^+ \longrightarrow \mathbf{N}$ *as follows:*

$$\mathrm{Reg}(k) = \begin{cases} 0 & \text{if } k = 1 \\ \sum_{i=1}^{k-1} \chi_{]0,\frac{1}{2}[}(\sigma_i) & \text{if } k \in \mathbf{N}^+ \setminus \{1\}, \end{cases}$$

where $\chi_{]0,\frac{1}{2}[}$ *is the characteristic function of the interval* $]0, \frac{1}{2}[$.

In order to make our central definition (3.7) easier to construct and understand, we also formulate the following lemma. The σ in the lemma works as a placeholder for the modified digitization parameters $\sigma_k^\wedge$.

Lemma 3.6. *If an irrational number σ fulfills $0 < \sigma < 1$ and $\delta = \mathrm{frac}\left(\frac{1}{\sigma}\right)$, then the value of $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ for natural $i \geqslant 2$ is the following:*

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \begin{cases} \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0,\frac{1}{2}[}(1-\delta) & \text{iff } \exists j \in \mathbf{N}^+ : \ \left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor \\ \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0,\frac{1}{2}[}(\delta) & \text{iff } \exists j \in \mathbf{N}^+ : \ i = \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor + 1. \end{cases}$$

This lemma introduces a recursive definition of digitization runs on all the digitization levels (Definition 3.7).

It is worth mentioning that the lemma determines the values of $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ for all natural $i \geqslant 2$. For example we get the value of $\left\lfloor \frac{2}{\sigma} \right\rfloor - \left\lfloor \frac{1}{\sigma} \right\rfloor$ (i.e., $i = 2$) by taking $j = 1$. Then, for each $j \geqslant 1$, number $\left\lfloor \frac{j}{\delta^\wedge} \right\rfloor + 1$ comes directly after $\left\lfloor \frac{j}{\delta^\wedge} \right\rfloor$ (the last value of i in the first line), while the next one, $\left\lfloor \frac{j}{\delta^\wedge} \right\rfloor + 2$, we get for $j+1$ as the first value of i in the first line. The lemma above thus implies that the values of $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ for $i = 2, 3, ...$ in this order are, if $\delta < \frac{1}{2}$: $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor - 1$ times $\left\lfloor \frac{1}{\sigma} \right\rfloor$, then one time $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$, then $\left\lfloor \frac{2}{\delta^\wedge} \right\rfloor - \left\lfloor \frac{1}{\delta^\wedge} \right\rfloor - 1$ times $\left\lfloor \frac{1}{\sigma} \right\rfloor$, then one time $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$, ..., $\left\lfloor \frac{j}{\delta^\wedge} \right\rfloor - \left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor - 1$ times $\left\lfloor \frac{1}{\sigma} \right\rfloor$, then one time $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$, and so on. If $\delta > \frac{1}{2}$, we only have to replace $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$ by $\left\lfloor \frac{1}{\sigma} \right\rfloor$ and $\left\lfloor \frac{1}{\sigma} \right\rfloor$ by $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$ in the above text.

Lemma 3.6 is a continuation of Lemma 3.1. Lemma 3.1 states that $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor$ for natural $i \geqslant 2$ can have one of two values $\left\lfloor \frac{1}{\sigma} \right\rfloor$ and $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$. Lemma 3.6 indicates exactly for which i we get each of the two values. It also shows with which frequencies both values appear. The frequencies are δ for the value $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$ and $1 - \delta$ for $\left\lfloor \frac{1}{\sigma} \right\rfloor$, where $\delta = \mathrm{frac}\left(\frac{1}{\sigma}\right)$ (see the discussion of Lemma 3.1). If $\delta < \frac{1}{2}$, the value $\left\lfloor \frac{1}{\sigma} \right\rfloor$ is the most frequent one, in case $\delta > \frac{1}{2}$ the most frequent one is $\left\lfloor \frac{1}{\sigma} \right\rfloor + 1$.

Because the phrase "the most frequent one" will become very important later in the text (see Proposition 3.12), we will discuss this in depth now. First, the sets of indices in the first line of the formula in Lemma 3.6 are nonempty for all $j \in \mathbf{N}^+$. More precisely, the sets of all consecutive indices $i \geqslant 2$ for which $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0,\frac{1}{2}[}(1-\delta)$ has the cardinality $\left\lfloor \frac{j}{\delta^\wedge} \right\rfloor - \left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor - 1$ for $j \in \mathbf{N}^+$, thus, because $0 < \delta^\wedge < 1$ is irrational, Lemma 3.6 can also be used for the calculation of these cardinalities and we get $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor$ or $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor - 1$ consecutive indices $i \geqslant 2$ for which $\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0,\frac{1}{2}[}(1-\delta)$ for $j \in \mathbf{N}^+$. Because $\delta^\wedge < \frac{1}{2}$, so $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor \geqslant 2$ and $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor - 1 \geqslant 1$ which gives the nonemptiness. The formula also ensures us that we get the value $\left\lfloor \frac{1}{\delta^\wedge} \right\rfloor \geqslant 2$ for some $j \geqslant 2$, namely for those $j \geqslant 2$ which are equal to $\left\lfloor \frac{k}{\theta^\wedge} \right\rfloor + 1$ for some $k \in \mathbf{N}^+$ if $\theta < \frac{1}{2}$ and for those $j \geqslant 2$ which are not equal to $\left\lfloor \frac{k}{\theta^\wedge} \right\rfloor + 1$ for any $k \in \mathbf{N}^+$ if $\theta > \frac{1}{2}$, where $\theta = \mathrm{frac}\left(\frac{1}{\delta^\wedge}\right)$. The phrase "the most frequent one" is thus well motivated.

Proof. Let $0 < \sigma < 1$ be any irrational number. For any natural number $i \geqslant 2$ we have:

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \frac{1}{\sigma} + \mathrm{frac}\left(\frac{i-1}{\sigma}\right) - \mathrm{frac}\left(\frac{i}{\sigma}\right).$$

As $\delta = \mathrm{frac}\left(\frac{1}{\sigma}\right)$ and σ is irrational, so also δ is irrational and $0 < \delta < 1$. Because $\mathrm{frac}\left(\frac{i}{\sigma}\right) = \mathrm{frac}\left(i \cdot \mathrm{frac}\left(\frac{1}{\sigma}\right)\right) = \mathrm{frac}(i\delta)$, we can proceed, using δ. Let's take any number $j \in \mathbf{N}^+$ and consider the following two cases:

[c.1.] a natural number $i \geqslant 2$ is such that $(i-1)\delta$ and $i\delta$ have the same value of the floor function, equal to $j-1$. For those i we have:

$$\operatorname{frac}\left(\tfrac{i-1}{\sigma}\right) = (i-1)\delta - (j-1) \quad \text{and} \quad \operatorname{frac}\left(\tfrac{i}{\sigma}\right) = i\delta - (j-1),$$

so we get

$$\left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor = \tfrac{1}{\sigma} - \operatorname{frac}\left(\tfrac{1}{\sigma}\right) = \left\lfloor \tfrac{1}{\sigma} \right\rfloor.$$

[c.2.] a natural number $i \geqslant 2$ is such that $(i-1)\delta$ and $i\delta$ have different values of the floor functions, equal to $j-1$ and j respectively (because $0 < \delta < 1$ and the integer parts of $(i-1)\delta$ and $i\delta$ are different in this case, they can only differ by 1). For those i we have:

$$\operatorname{frac}\left(\tfrac{i-1}{\sigma}\right) = (i-1)\delta - (j-1) \quad \text{and} \quad \operatorname{frac}\left(\tfrac{i}{\sigma}\right) = i\delta - j,$$

so we get

$$\left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor = \tfrac{1}{\sigma} + 1 - \operatorname{frac}\left(\tfrac{1}{\sigma}\right) = \left\lfloor \tfrac{1}{\sigma} \right\rfloor + 1.$$

In order to prove the lemma for $\delta > \frac{1}{2}$, we observe the following:

Remark. *Let $\delta \in]\frac{1}{2}, 1[$. For all $j \in \mathbf{N}^+$ and natural $i > j$:*

$$[i\delta < i - j \quad \Leftrightarrow \quad i\delta^\wedge > j] \quad \textit{and} \quad [i\delta > i - j \quad \Leftrightarrow \quad i\delta^\wedge < j].$$

To prove this it is enaugh to notice that $\delta^\wedge = 1 - \delta$ for $\delta \in]\frac{1}{2}, 1[$.

Because $\delta^\wedge$ is irrational for all δ as described in the lemma, we have:

$$\begin{aligned}
&\exists j \in \mathbf{N}^+ : \ \left\lfloor \tfrac{j-1}{\delta^\wedge} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \tfrac{j}{\delta^\wedge} \right\rfloor \\
\Leftrightarrow \quad & \exists j \in \mathbf{N}^+ : \ j - 1 < (i-1)\delta^\wedge < i\delta^\wedge < j \\
\overset{(1)}{\Leftrightarrow} \quad & \begin{cases} \exists j \in \mathbf{N}^+ : \ j - 1 < (i-1)\delta < i\delta < j & \text{if } \delta < \frac{1}{2} \\ \exists j \in \mathbf{N}^+ : \ [i - j - 1 < (i-1)\delta < i - j & \\ \wedge \quad i - j < i\delta < i - j + 1] & \text{if } \delta > \frac{1}{2} \end{cases} \\
\Leftrightarrow \quad & \begin{cases} \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor & \text{if } \delta < \frac{1}{2} \\ \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor + 1 & \text{if } \delta > \frac{1}{2} \end{cases} \\
\overset{(2)}{\Leftrightarrow} \quad & \begin{cases} \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor & \text{if } \delta < \frac{1}{2} \\ \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1 & \text{if } \delta > \frac{1}{2} \end{cases} \\
\Leftrightarrow \quad & \left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor = \left\lfloor \tfrac{1}{\sigma} \right\rfloor + \chi_{]0, \frac{1}{2}[}(1 - \delta),
\end{aligned}$$

which proves the first statement in the lemma. Equivalence (1) we get using the above remark for $\delta > \frac{1}{2}$, equivalence (2) using [c.1.] and [c.2.]. The fact that $0 < \delta^\wedge < \frac{1}{2}$ (which means that $\frac{1}{\delta^\wedge} > 2$) assures that the set of such i that $\left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor$ is not empty for all $j \in \mathbf{N}^+$.

An analogous reasoning can be made for the second statement in the lemma:

$$\exists j \in \mathbf{N}^+ : \quad i = \left\lfloor \tfrac{j}{\delta^\wedge} \right\rfloor + 1$$

$$\Leftrightarrow \qquad \exists j \in \mathbf{N}^+ : \quad [j-1 < (i-1)\delta^\wedge < j \quad \wedge \quad j < i\delta^\wedge < j+1]$$

$$\Leftrightarrow \qquad \begin{cases} \exists j \in \mathbf{N}^+ : \quad [j-1 < (i-1)\delta < j & \\ \wedge \quad j < i\delta < j+1] & \text{if } \delta < \frac{1}{2} \\ \exists j \in \mathbf{N}^+ : \quad i-j-1 < (i-1)\delta < i\delta < i-j & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \qquad \begin{cases} \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor + 1 & \text{if } \delta < \frac{1}{2} \\ \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \qquad \begin{cases} \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1 & \text{if } \delta < \frac{1}{2} \\ \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \qquad \left\lfloor \tfrac{i}{\sigma} \right\rfloor - \left\lfloor \tfrac{i-1}{\sigma} \right\rfloor = \left\lfloor \tfrac{1}{\sigma} \right\rfloor + \chi_{]0,\frac{1}{2}[}(\delta).$$

The lemma is proved. □

The next definition is the basis for the theorem describing digital straight lines with irrational slope:

Definition 3.7. *For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the following functions:*

- $\mathrm{run}_1 : \mathbf{N}^+ \to \mathcal{P}(\mathbf{N}^+)$, *defined as follows:* $\mathrm{run}_1(j) = \{i;\ \left\lfloor \frac{j-1}{a} \right\rfloor + 1 \leqslant i \leqslant \left\lfloor \frac{j}{a} \right\rfloor\}$ *for* $j \in \mathbf{N}^+$
- *For* $k \in \mathbf{N}^+ \setminus \{1\}$: $\quad \mathrm{run}_k : \mathbf{N}^+ \to \mathcal{P}(\mathrm{run}_{k-1}(\mathbf{N}^+))$ *defined as follows:* $\mathrm{run}_k(1) = \{\mathrm{run}_{k-1}(i);\ 1 \leqslant i \leqslant \left\lfloor \frac{1}{\sigma^\wedge_{k-1}} \right\rfloor + \mathrm{Rmod}_2(k)\}$, *and for natural* $j \geqslant 2$*:* $\mathrm{run}_k(j) = \{\mathrm{run}_{k-1}(i);\ \left\lfloor \frac{j-1}{\sigma^\wedge_{k-1}} \right\rfloor + \mathrm{Rmod}_2(k) + 1 \leqslant i \leqslant \left\lfloor \frac{j}{\sigma^\wedge_{k-1}} \right\rfloor + \mathrm{Rmod}_2(k)\}$, *where*

$$\mathrm{Rmod}_2(k) = \begin{cases} 0 & \textit{if } \mathrm{Reg}(k) \textit{ is even} \\ 1 & \textit{if } \mathrm{Reg}(k) \textit{ is odd,} \end{cases}$$

$\sigma^\wedge_k$ *are the modified digitization parameters defined in Definition 3.4, the function* Reg *is defined in Definition 3.5 and* $\mathcal{P}(A)$ *denotes the power set of a set* A*.*

We shall say that $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ is a *run of digitization level* k. We will also write run_k or in plural runs_k, meaning $\mathrm{run}_k(j)$ *for some* $j \in \mathbf{N}^+$, or, respectively, $\{\mathrm{run}_k(i);\ i \in I\}$ *where* $I \in \mathcal{P}(\mathbf{N}^+)$. Also here we define the *length of a digitization run* as its cardinality.

From the definition of run_1 it is clear that runs_1 can be identified with digitization runs described in the beginning of this section, because for $j \in \mathbf{N}^+$

(according to Proposition 3.2):

$$\begin{aligned}\operatorname{run}_1(j) &= \{i \in \mathbf{N}^+;\ \left\lfloor \tfrac{j-1}{a} \right\rfloor + 1 \leqslant i \leqslant \left\lfloor \tfrac{j}{a} \right\rfloor\} = \\ &= \{i \in \mathbf{N}^+;\ j-1 < ai \leqslant j\} = \\ &= \{i \in \mathbf{N}^+;\ (i,j) \in\ D_{R'}(l)\},\end{aligned}$$

while

$$\operatorname{run}(j) = \{(i,j) \in (\mathbf{N}^+)^2;\ \ (i,j) \in D_{R'}(l)\}.$$

Proposition 3.8. *Let l be given by the equation $y = ax$ where $0 < a < 1$ and a is irrational. For each $k \in \mathbf{N}^+ \setminus \{1\}$, the runs of the level k can have one of the two lengths: $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor$ (short runs) or $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor + 1$ (long runs). The runs of level 1 can have lengths $\left\lfloor \frac{1}{a} \right\rfloor$ or $\left\lfloor \frac{1}{a} \right\rfloor + 1$.*

Proof. For level k where $k \in \mathbf{N}^+ \setminus \{1\}$ the length of $\operatorname{run}_k(1)$ is equal to $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor$ if $\operatorname{Rmod}_2(k) = 0$ and $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor + 1$ if $\operatorname{Rmod}_2(k) = 1$. If $j \geqslant 2$ is a natural number, then the length of $\operatorname{run}_k(j)$ is equal to $\left\lfloor \frac{j}{\sigma^{\wedge}_{k-1}} \right\rfloor - \left\lfloor \frac{j-1}{\sigma^{\wedge}_{k-1}} \right\rfloor$ and, because $0 < \sigma^{\wedge}_i < 1$ for $i \in \mathbf{N}^+$, we can apply Lemma 3.6 with $\sigma = \sigma^{\wedge}_{k-1}$ for $k = 2, 3, \ldots$. For $k = 1$ we apply Lemma 3.6 with $\sigma = a$. □

Lemma 3.6 and Proposition 3.8 allow us to formulate the following definitions:

Definition 3.9. *For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the following functions for $k \in \mathbf{N}^+$:*

$$\operatorname{kind\_run}_k : \mathbf{N}^+ \to \{S, L\},$$

where 'S' and 'L' are abbreviations for short *and* long *respectively. For $j \in \mathbf{N}^+$:*

$$\operatorname{kind\_run}_1(j) = \begin{cases} S & \text{if } \operatorname{card}(\operatorname{run}_1(j)) = \left\lfloor \frac{1}{a} \right\rfloor \\ L & \text{if } \operatorname{card}(\operatorname{run}_1(j)) = \left\lfloor \frac{1}{a} \right\rfloor + 1 \end{cases}$$

$$\operatorname{kind\_run}_k(j) = \begin{cases} S & \text{if } \operatorname{card}(\operatorname{run}_k(j)) = \left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor \\ L & \text{if } \operatorname{card}(\operatorname{run}_k(j)) = \left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor + 1 \end{cases} \quad \text{for } k \in \mathbf{N}^+ \setminus \{1\},$$

where $\operatorname{card}(\operatorname{run}_k(j))$ denotes the number of elements in $\operatorname{run}_k(j)$ (the length *of $\operatorname{run}_k(j)$).*

Definition 3.10. *For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the* alternation-function

$$\operatorname{alt} : \{S, L\} \to \{S, L\}$$

as follows:

$$\operatorname{alt}(S) = L, \ \ \operatorname{alt}(L) = S.$$

We define three functions with level numbers as arguments:

Definition 3.11. *For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define three functions:*

$$\mathrm{single}_{(.)},\ \mathrm{main}_{(.)},\ \mathrm{first}_{(.)} : \mathbf{N}^+ \to \{S, L\}$$

For $k \in \mathbf{N}^+$:

$$\mathrm{single}_k = \begin{cases} S & \text{if } \{j \in \mathbf{N}^+;\ \mathrm{kind\_run}_k(j) = \mathrm{kind\_run}_k(j+1) = S\} = \emptyset \\ L & \text{if } \{j \in \mathbf{N}^+;\ \mathrm{kind\_run}_k(j) = \mathrm{kind\_run}_k(j+1) = L\} = \emptyset \end{cases}$$

$\mathrm{main}_k = \mathrm{alt} \circ \mathrm{single}_k$

$\mathrm{first}_k = \mathrm{kind\_run}_k(1)$.

We remark that the kth digitization parameter defined in Definition 3.4 has the following influence on the most frequent (main) run length on the level k:

Proposition 3.12. *For a digital line $y = ax$, where $0 < a < 1$ and a is irrational, we have on the level k where $k \in \mathbf{N}^+$:*

- $\sigma_k < \frac{1}{2} \ \Rightarrow\ \mathrm{main}_k = S$,
- $\sigma_k > \frac{1}{2} \ \Rightarrow\ \mathrm{main}_k = L$.

Proof. Combine Proposition 3.8 with the discussion after the statement of Lemma 3.6. □

This brings us to the following theorem:

Theorem 3.13. *[Necessary condition to be a digital line with irrational slope]. For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, the R'-digitization of the positive half line of l is the following subset of $\mathbf{Z}^2$:*

$$D_{R'}(l) = \bigcup_{j \in \mathbf{N}^+} \{\mathrm{run}_1(j) \times \{j\}\}.$$

For each $k \in \mathbf{N}^+$ runs of the level k defined in Definition 3.7 fulfill the following conditions:

[N1]: *There are only two possible run-lengths on the level k. They are expressed by two consecutive natural numbers. The length of* $\mathrm{run}_k(j)$ *for $j \in \mathbf{N}^+ \setminus \{1\}$ is namely $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor$ ($\left\lfloor \frac{1}{a} \right\rfloor$ if $k = 1$) or $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor + 1$ ($\left\lfloor \frac{1}{a} \right\rfloor + 1$ if $k = 1$), where $\sigma^{\wedge}_k$ is the modified digitization parameter defined in Definition 3.4. We write* $\mathrm{kind\_run}_k(j) = S$ *or* $\mathrm{kind\_run}_k(j) = L$ *respectively. S and L are abbreviations of* short *and* long *respectively.*

[N2]: $\mathrm{kind\_run}_k(\left\lfloor \frac{j}{\sigma^{\wedge}_k} \right\rfloor + 1) = \mathrm{single}_k$ *for all $j \in \mathbf{N}^+$ and* $\mathrm{kind\_run}_k(i) = \mathrm{main}_k$ *for all natural $i \geqslant 2$ such that $i \neq \left\lfloor \frac{j}{\sigma^{\wedge}_k} \right\rfloor + 1$ for all $j \in \mathbf{N}^+$.* single_k *means the kind of* run_k *which can never appear more than once in a sequence and* main_k *means the kind of* run_k *which comes in multiples.*

[N3]: *The kind of the first run of the level k is determined by the following formula:*

$$\mathrm{first}_k = \mathrm{kind\_run}_k(1) = \begin{cases} S & \text{if } \mathrm{Reg}(k) \text{ is even} \\ L & \text{if } \mathrm{Reg}(k) \text{ is odd}\ , \end{cases}$$

where the function Reg *is defined in Definition 3.5.*

Proof. Let us first consider the case $k = 1$. Because runs_1 can be identified with digitization runs described in the beginning of this section, Proposition 3.2 and Lemma 3.6 with $\sigma = a$ prove the conditions [N1], [N2] and [N3] for level 1.

The case $k > 1$ remains to be considered. From Definition 3.4 follows that we can apply Lemma 3.6 to $\sigma = \sigma_{k-1}^{\wedge}$ (so $\delta = \sigma_k$) for $k = 2, 3, ...$ This lemma proves by simple induction the conditions [N1] and [N2], because runs_{k-1} are the elements of runs_k.
It remains to prove the condition [N3]. First we assume that for the digitization parameters of the line to digitize the following holds: $\sigma_k < \frac{1}{2}$ for all $k \in \mathbf{N}^+$ and we prove the condition [N3] for lines like this. If $j \in \mathbf{N}^+$, $\text{runs}_k(i)$ $(i \geqslant 2)$ belonging to the $\text{run}_{k+1}(j)$ are short (i.e., have length $\left\lfloor \frac{1}{\sigma_{k-1}} \right\rfloor$) if and only if

$$\left\lfloor \frac{j-1}{\sigma_k} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\sigma_k} \right\rfloor$$

(Lemma 3.6 with $\sigma = \sigma_{k-1}$), so the $\text{run}_{k+1}(j)$ consists of $\left\lfloor \frac{j}{\sigma_k} \right\rfloor - \left\lfloor \frac{j-1}{\sigma_k} \right\rfloor - 1$ short runs_k and one long, $\text{run}_k(\left\lfloor \frac{j-1}{\sigma_k} \right\rfloor + 1)$ or $\text{run}_k(\left\lfloor \frac{j}{\sigma_k} \right\rfloor + 1)$. In particular, for $j = 1$ we get that $\text{run}_{k+1}(1)$ consists of $\left\lfloor \frac{1}{\sigma_k} \right\rfloor - 1$ short runs_k (numbers $2, ..., \left\lfloor \frac{1}{\sigma_k} \right\rfloor$) and we know (Lemma 3.6) that $\text{run}_k(\left\lfloor \frac{1}{\sigma_k} \right\rfloor + 1)$ is long, so $\text{run}_{k+1}(1)$ is:

- **short** if $\text{run}_k(1)$ is long,
- **long** if $\text{run}_k(1)$ is short.

Because the first run of the level 1 (first_1) is always short, we get by simple induction the following statement for the lines with all digitization parameters less than $\frac{1}{2}$: For $k \in \mathbf{N}^+$:

$$\text{first}_k = \begin{cases} S & \text{if } k \text{ is odd} \\ L & \text{if } k \text{ is even.} \end{cases}$$

We can also say that for the lines as described above: *the kind of the first run is alternating for consecutive levels.* From Definition 3.5 follows that for the lines with all the digitization parameters $\sigma_1, \sigma_2, ...$ less than $\frac{1}{2}$ we have $\text{Reg}(k) = k - 1$ for $k \in \mathbf{N}^+$, thus its value is odd for even k and even for odd k. This shows that the statement above is equivalent to the condition [N3] for the lines with $\sigma_k < \frac{1}{2}$ for all $k \in \mathbf{N}^+$ and the proof of the theorem for this type of lines is complete.
If $\sigma_k > \frac{1}{2}$ for some $k \in \mathbf{N}^+$ then we get by the same reasoning as above (Lemma 3.6 with $\sigma = a$ if $k = 1$ and $\sigma = \sigma_{k-1}^{\wedge}$ if $k > 1$, thus $\delta = \sigma_k$ and $1 - \delta = \sigma_k^{\wedge}$) that $\text{run}_{k+1}(1)$ consists of $\left\lfloor \frac{1}{\sigma_k^{\wedge}} \right\rfloor - 1$ long runs_k (numbers $2, ..., \left\lfloor \frac{1}{\sigma_k^{\wedge}} \right\rfloor$) and we know (also Lemma 3.6) that $\text{run}_k(\left\lfloor \frac{1}{\sigma_k^{\wedge}} \right\rfloor + 1)$ is short, so $\text{run}_{k+1}(1)$ is:

- **long** if $\text{run}_k(1)$ is long,

- **short** if $\mathrm{run}_k(1)$ is short.

and the alternation pattern breaks. We get no alternation of the kind of the first run from level k to level $k+1$ if $\sigma_k > \frac{1}{2}$, and a simple induction proof gives us the following recurrent description of the kind of the first run on each level:

- $\mathrm{first}_1 = S$,
- For each natural $k \geqslant 2$: if $\sigma_{k-1} < \frac{1}{2}$, then $\mathrm{first}_k = \mathrm{alt} \circ \mathrm{first}_{k-1}$ (where $\mathrm{alt}(S) = L$ and $\mathrm{alt}(L) = S$ according to Definition 3.10),
- For each natural $k \geqslant 2$: if $\sigma_{k-1} > \frac{1}{2}$, then $\mathrm{first}_k = \mathrm{first}_{k-1}$,

which, according to Definition 3.5, leads to the condition [N3] in Theorem 3.13. The proof is now complete. □

Generally speaking, we have two important questions in connection with digital lines:

- how to find the digitization of a given real line (necessary condition to be a digital line)
- how to recognize a digital line in a subset of $\mathbf{Z}^2$ (sufficient condition to be a digital line).

To give a simple answer to the first question, we will reformulate the results from Theorem 3.13 in a more practically useful way. To do this, we will use function Reg to describe the form of runs on each digitization level. The form of runs on level $k+1$ depends on both main (thus on σ_k in a very explicit way) and first on the level k (the first on the level k for $k \geqslant 2$ is fully determined only by the digitization parameters $\sigma_1, ..., \sigma_{k-1}$. They show where the kind of the first run alternates from one level to the next level and where not).
It can be convenient to use the symbols $S \cdots SL$, $LS \cdots S$, $L \cdots LS$ and $SL \cdots L$ when describing the form of digitization runs. For example $S \cdots SL$ will mean that the run_k we are talking about consists of $\lfloor 1/\sigma^{\wedge}_{k-1} \rfloor - 1$ or $\lfloor 1/\sigma^{\wedge}_{k-1} \rfloor$ short runs_{k-1} (abbrev. S) and one long run_{k-1} (abbrev. L) in this order, so it is a run with main element short.

Corollary 3.14. *[Necessary condition to be digital line with irrational slope]. For a straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we have:*
for each $j \in \mathbf{N}^+$, $\mathrm{run}_1(j)$ can have two possible lengths: $\lfloor \frac{1}{a} \rfloor$ (S - short) and $\lfloor \frac{1}{a} \rfloor + 1$ (L - long) and the forms of runs_{k+1} ($\mathrm{form\_run}_{k+1}$) *for $k \in \mathbf{N}^+$ are as follows:*

$$\mathrm{form\_run}_{k+1} = \begin{cases} S \cdots SL & \textit{iff } \mathrm{Reg}(k+1) = \mathrm{Reg}(k) + 1, \quad \mathrm{Reg}(k) \ \textit{is even} \\ SL \cdots L & \textit{iff } \mathrm{Reg}(k+1) = \mathrm{Reg}(k), \quad \mathrm{Reg}(k) \ \textit{is even} \\ LS \cdots S & \textit{iff } \mathrm{Reg}(k+1) = \mathrm{Reg}(k) + 1, \quad \mathrm{Reg}(k) \ \textit{is odd} \\ L \cdots LS & \textit{iff } \mathrm{Reg}(k+1) = \mathrm{Reg}(k), \quad \mathrm{Reg}(k) \ \textit{is odd}\,, \end{cases}$$

where S means run_k *with length* $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor$ *and L means* run_k *with length* $\left\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \right\rfloor + 1$ *and the function* Reg *is defined in Definition 3.5.*

Proof. This corollary follows from Definition 3.7 and Theorem 3.13. We have two implications: $\sigma_k < \frac{1}{2} \Rightarrow \text{main}_k = S$ and $\sigma_k > \frac{1}{2} \Rightarrow \text{main}_k = L$ (Proposition 3.12). The parity of $\text{Reg}(k)$ determines the first run of the level k (first_k is short if $\text{Reg}(k)$ is even and long if $\text{Reg}(k)$ odd - Condition [N3]).

The reasoning of the proof is illustrated in the following table; assumptions in the first two columns, conclusions, which are based on the above statements, in the three last columns:

σ_k	$\text{Reg}(k)$	main_k	first_k	form of run_{k+1}
$< \frac{1}{2}$	even	S	S	$S \cdots SL$, $\left\lfloor \frac{1}{\sigma_k} \right\rfloor - 1$ or $\left\lfloor \frac{1}{\sigma_k} \right\rfloor$ times 'S'
$> \frac{1}{2}$	even	L	S	$SL \cdots L$, $\left\lfloor \frac{1}{1-\sigma_k} \right\rfloor - 1$ or $\left\lfloor \frac{1}{1-\sigma_k} \right\rfloor$ times 'L'
$< \frac{1}{2}$	odd	S	L	$LS \cdots S$, $\left\lfloor \frac{1}{\sigma_k} \right\rfloor - 1$ or $\left\lfloor \frac{1}{\sigma_k} \right\rfloor$ times 'S'
$> \frac{1}{2}$	odd	L	L	$L \cdots LS$, $\left\lfloor \frac{1}{1-\sigma_k} \right\rfloor - 1$ or $\left\lfloor \frac{1}{1-\sigma_k} \right\rfloor$ times 'L'

The relation of the parities of $\text{Reg}(k)$ and $\text{Reg}(k+1)$ determines the main of the level k:

- if $\text{Reg}(k+1)$ and $\text{Reg}(k)$ have the same parities, then $\chi_{]0,\frac{1}{2}[}(\sigma_k) = 0$, so $\sigma_k > \frac{1}{2}$ and main of the level k is long.
- if $\text{Reg}(k+1)$ and $\text{Reg}(k)$ have different parities, then $\chi_{]0,\frac{1}{2}[}(\sigma_k) = 1$, so $\sigma_k < \frac{1}{2}$ and main of the level k is short.

Because runs_k are elements of the runs_{k+1}, the conclusion about the form of the runs of the level $k+1$ follows from the information above. □

The corollary is constructive. It shows exactly how to find the R′-digitization of the positive half line $y = ax$ (where $0 < a < 1$ and a is irrational). We get the digitization by calculating the digitization parameters and proceeding step by step, following the recursive description. The knowledge about the kind of the first run on each level allows us go as far as we want in the digitization.

Corollary 3.14 shows a necessary condition for a subset of $(\mathbf{N}^+)^2$ to be a digital (half) line. Now the question remains whether the condition is also sufficient. We can ask ourselves whether all the subsets of $(\mathbf{N}^+)^2$ fulfilling on all the levels the three conditions named in Theorem 3.13 and with the short run length on the level k equal to $n_k \geqslant 2$ are digitizations of some (half) lines with irrational slope. In other words: can all the sequences of natural numbers greater or equal to 2 be the short run lengths for some line? Run length 1 on level with number greater than 1 is only possible for lines with rational slope, where we get periodical digitization, so there is only one kind of run on some level k, where $k \in \mathbf{N}^+$ depends on slope. If the slope is irrational, we can only have short run length 1 on the level 1, i.e., only short run_1 can have length 1.

Lemma 3.15. *For each $k \in \mathbf{N}^+$:*

- *For each $0 < r < 1$ it is possible to find a real straight line with the level k parameter $\sigma_k = r$.*
- *If $k \geqslant 2$: for each $0 < r < 1$ and each set $\{i_1, ..., i_l\} \subset \{1, ..., k-1\}$ with cardinality $1 \leqslant l \leqslant k-1$ it is possible to find a real straight line with the level k parameter $\sigma_k = r$ and such that $\sigma_i > \frac{1}{2}$ for all $i \in \{i_1, ..., i_l\}$.*

Proof. We construct the slope of the line $y = ax$ fulfilling this condition as follows:

- In the first case we take $a = [0, n_1, ..., n_{k-1}, [n_k, r]]$ where $n_1 \geqslant 1$ and $n_i \geqslant 2$ for $i \geqslant 2$ are natural numbers. $[0, n_1, ..., n_{k-1}, [n_k, r]]$ is a compact abbreviated form of the continued fraction (see Hardy & Wright (1979), p.130.):

$$\cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots + \cfrac{1}{n_{k-1} + \cfrac{1}{n_k + r}}}}$$

  Then $n_i = \left\lfloor \frac{1}{\sigma_{i-1}} \right\rfloor$ for $i = 2, ..., k$ is the length of short run$_i$ and $n_1 = \left\lfloor \frac{1}{a} \right\rfloor$ is the length of short run$_1$. All the straight lines with the slopes a like above fulfill the imposed condition. In each case we have $\sigma_k = \mathrm{frac}(n_k + r) = r$. The restriction $n_i \geqslant 2$ for $i \geqslant 2$ ensures that all the σ_i for $i = 1, ..., k-1$ are less than $\frac{1}{2}$, so we never have to modify the digitization parameters according to Definition 3.4, and we really get $\sigma_k = \mathrm{frac}(n_k + r) = r$.

- In the second case we do similarily as in the proof of the first part of the lemma. If we wish to have $\sigma_i > \frac{1}{2}$, then we put 1 twice in place of n_{i+1} in the continued fraction, i.e., we replace

$$[0, n_1, ..., n_i, n_{i+1}, n_{i+2}, ..., n_{k-1}, [n_k, r]]$$

  by

$$[0, n_1, ..., n_i, 1, n_{i+1} - 1, n_{i+2}, ..., n_{k-1}, [n_k, r]].$$

  In other words, we put

$$1 + \frac{1}{n_{i+1} - 1 + \cdots}$$

  in the continued fraction in place of '$n_{i+1}+$'. This we can repeat on each of the levels with numbers $i \in \{i_1, ..., i_l\} \subset \{1, ..., k-1\}$. Each digitization level i with $\sigma_i > \frac{1}{2}$ causes increasing (by one) of the number of levels (literally) in the continued fraction which is going to be the slope. The construction of the slope is based purely on Definition 3.4.

The proof is now complete. □

This leads to the following theorem:

Theorem 3.16. *Let $n \in \mathbf{N}^+$. For each sequence of natural numbers $(k_1, k_2, ..., k_n)$ such that $k_1 \geqslant 1$ and $k_i > 1$ for $1 < i \leqslant n$ there exist m lines $y = ax$ with rational slopes, where*

$$m = \begin{cases} 2^{n-1} & \text{if} \quad k_n \neq 2 \\ 2^{n-2} & \text{if} \quad k_n = 2 \end{cases}$$

and their digitization fulfills the following conditions:
for $i = 1, ..., n$ the short run's length on digitization level i is k_i.

Proof. For a sequence $(k_1, k_2, ..., k_n)$ fulfilling the assumptions named in the theorem, we define the slopes of the lines as follows: $a = [0, k_1, ..., k_n]$ (continued fraction $[0, k_1, ..., k_{n-1}, [k_n, 0]]$ as defined in the proof of Lemma 3.15) if we want all the $\sigma_j < \frac{1}{2}$ for $j = 1, ..., n-1$. If we want $\sigma_i > \frac{1}{2}$ for some $1 \leqslant i \leqslant n-1$, we take $a = [0, k_1, ...k_i, 1, k_{i+1} - 1, k_{i+2}, ..., k_n]$. We have to make a decision about $\sigma_i < \frac{1}{2}$ or $\sigma_i > \frac{1}{2}$ for $i = 1, ..., n-1$ which means in $n-1$ places. This gives us 2^{n-1} possibilities. We have $\sigma_{n-1} = \frac{1}{k_n}$, so, if $k_n = 2$, then $\sigma_{n-1} = \frac{1}{2}$ and we have one place less to make a choice, so we have only 2^{n-2} possibilities. It follows from Theorem 3.13 that the lines with those slopes fulfill the desired condition about the short runs' lengths. □

Theorem 3.16 states that all sequences of natural numbers greater or equal to 2 (and the first element possibly equal to 1) generate the digitization of some lines with short runs' lengths on each level defined by the elements of the sequence. This means that each construction of pixels as described in Theorem 3.13, with infinitely many (n was arbitrary!) digitization levels is the R′-digitization of the positive half line of some line $y = ax$, where $0 < a < 1$ is irrational. This gives the following theorem, which states that the necessary condition for being a digital line with irrational slope $0 < a < 1$ is also sufficient:

Theorem 3.17. *[Sufficient condition to be a digital line with irrational slope]. Each subset of $(\mathbf{N}^+)^2$ containing $(1, 1)$ and fulfilling the conditions* [N1], [N2] *and* [N3] *on all the levels is the R′-digitization of the positive half line of some line $y = ax$, where $0 < a < 1$ and a is irrational.*

Continued fractions have already been used in this context; Rosenfeld and Klette indicate in their paper two independent publications from 1991: one by M. Bruckstein and another one by K. Voss; see Rosenfeld & Klette (2001).

4. Conclusions

We have formulated a formal definition of digitization runs and theorems containing necessary and sufficient conditions for subsets of $(\mathbf{N}^+)^2$ being the digitization of a straight (half) line with irrational slope passing through the origin. Only methods of elementary mathematics have been applied. The main topic of interest was Theorem 3.13 with the necessary condition. The restrictions put on the line (irrational slope $0 < a < 1$ and digitization of the positive half line only) are not severe restrictions. It is not difficult to expand the theory to the cases not explicitly covered in this paper. The developed and proved theory can be used in the research into the theory of digital lines, their symmetries, translations etc.

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