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Hanna Uscka-Wehlou

Some combinatorial problems related to digital straight lines with irrational slopes and to balanced aperiodic words

KTH, 2 December 2009

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Items:

Mechanical words and digital lines

A short introduction to continued fractions

Some combinatorics on continued fraction elements

Questions and problems

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Mechanical words and digital lines

A short introduction to continued fractions

Some combinatorics on continued fraction elements

Questions and problems

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1

Words and lines

1.1

Words

Finite words

$\mathcal{A}$ – alphabet (a set of symbols)

$\mathcal{A}^*$ - the set of finite words over $\mathcal{A}$

$(\mathcal{A}^*, +)$ - is a monoid :

- concatenation (+) is associative $(u+v)+w=u+(v+w)$

  101010+1111=1010101111

- the empty word ε is the neutral element

$(\mathcal{A}^*, +)$ is called the free monoid on the set $\mathcal{A}$.

- no inverse operation, no commutativity

Infinite words

A – alphabet (a set of symbols)

A $^{\omega}$ - the set of right infinite words over *A*

For example, if *A*={1,2}, then the words are:

$$w\colon \mathbf{N}^{+} \to \{1,2\}$$

$$w = w(1)w(2)w(3)\cdots \in \{1,2\}^{\omega}$$

The word w is called a factor of a word u if there exist words x, y such that $u=x+w+y$.

1222 is a factor of 0001222111132131101010001

10101 is a factor of 10101010101010101

ABCDA is a factor of CBBBDACADBCAABCDA

10101 is a factor of 10101

Sturmian words

Sturmian words are infinite words which have exactly $m+1$ different factors of length m for every natural m.

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m=4

1010, 0101, 0010, 1001, 0100.

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Balanced words (binary)

n – the length of the word

m – any positive natural number less than n

Each m-letter long factor of this word can contain either k or $k+1$ 1's

An example:

$n = 41$

$m = 16$

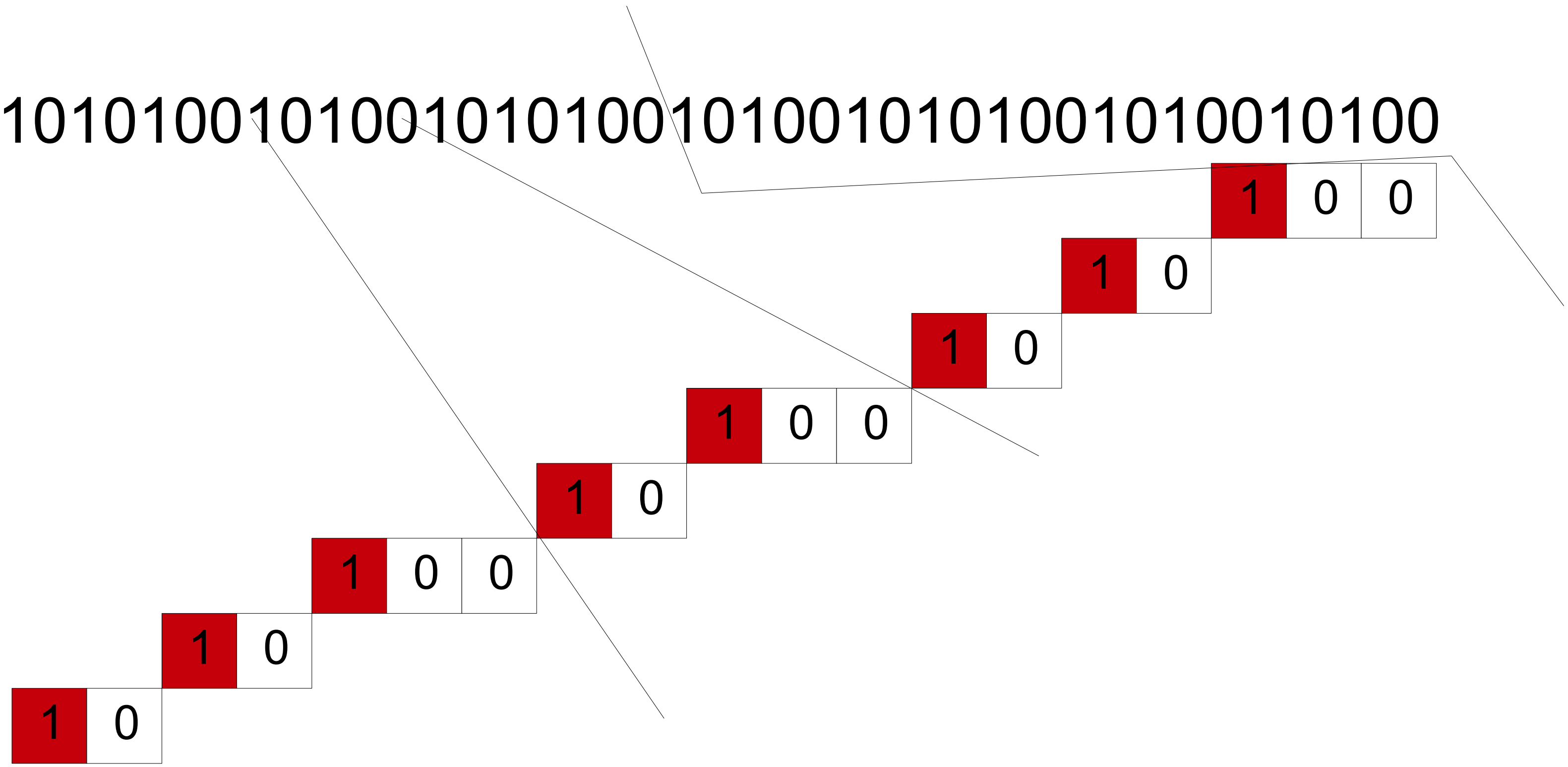
10101001010010101001010010101001010010100

7 6

$k = 6$

Balanced words give straight lines

10101001010010101001010010101001010010100



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Upper and lower mechanical, characteristic words

$$s'(a), s(a)\colon \mathbf{N} \to \{0,1\}$$

$$\forall\, n \in \mathbf{N} \qquad s'_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil,$$

$$s_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor$$

$$c(a)\colon \mathbf{N}^+ \to \{0,1\}$$

$$\forall\, n \in \mathbf{N}^+ \qquad c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor$$

Sturmian words : different characterizations

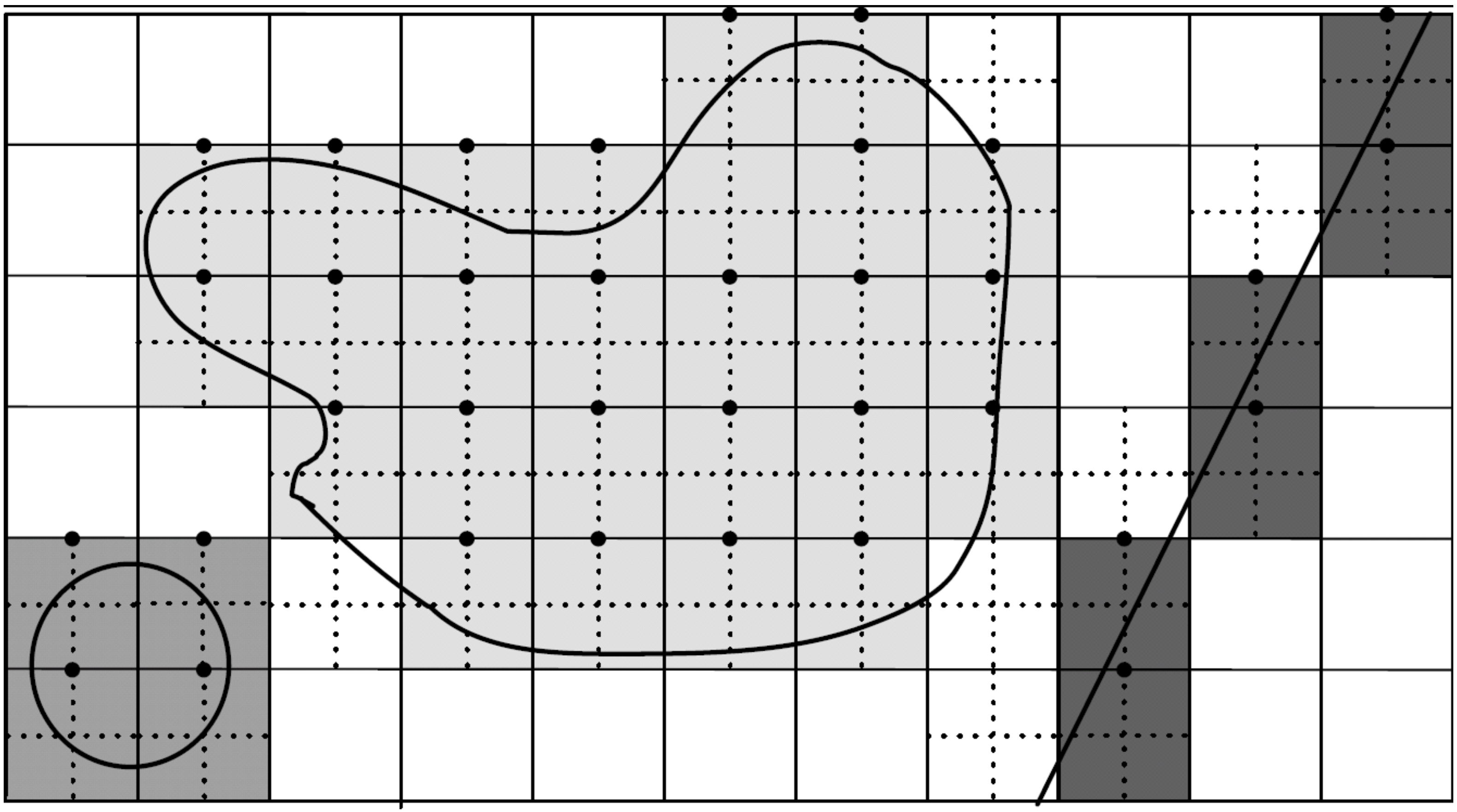
Theorem Let s be an infinite word.
The following are equivalent:

- s is Sturmian;
- s is balanced and aperiodic;
- s is irrational (lower or upper) mechanical.

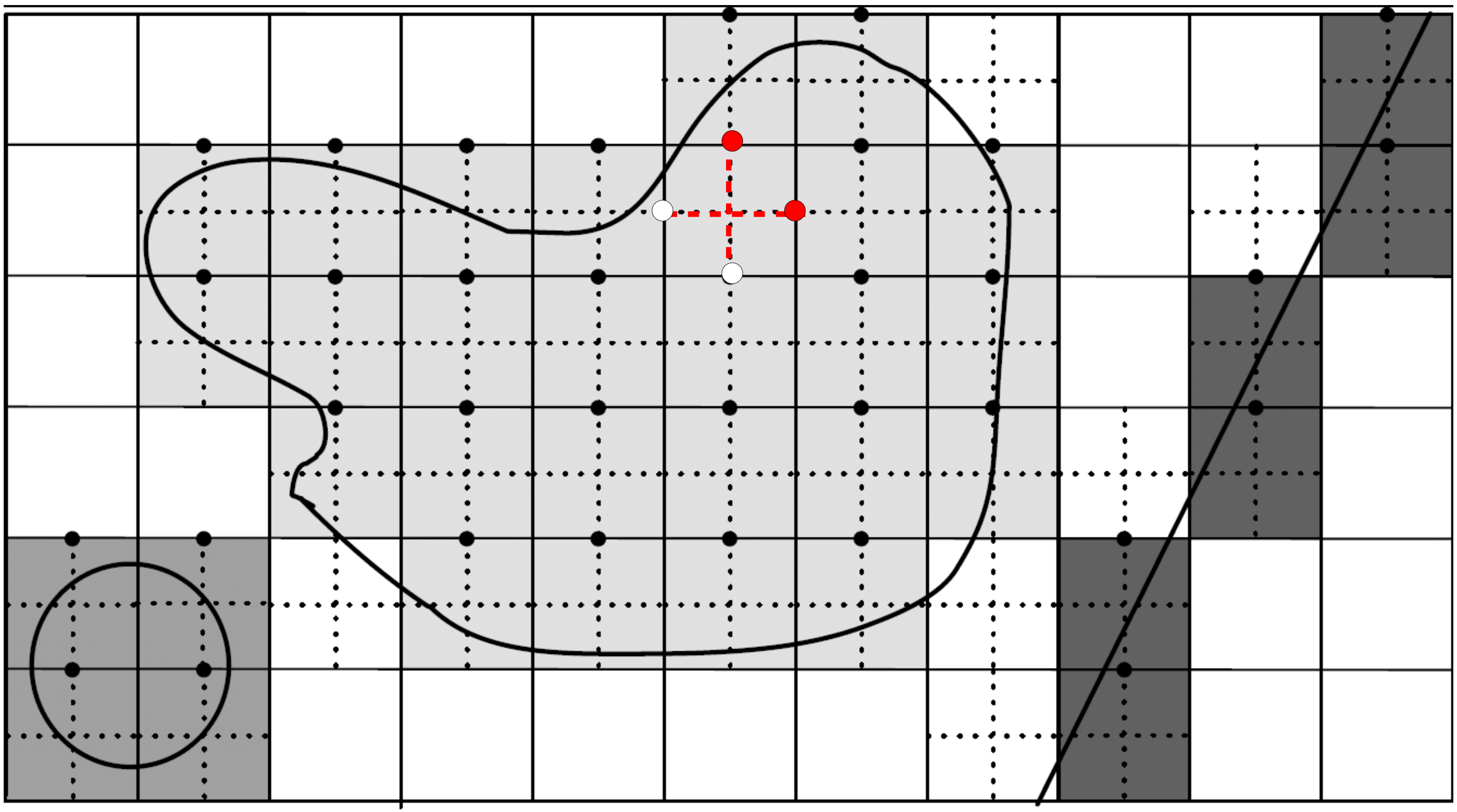
1.2

Lines

Digital geometry – R'-digitization



Digital geometry – R'-digitization

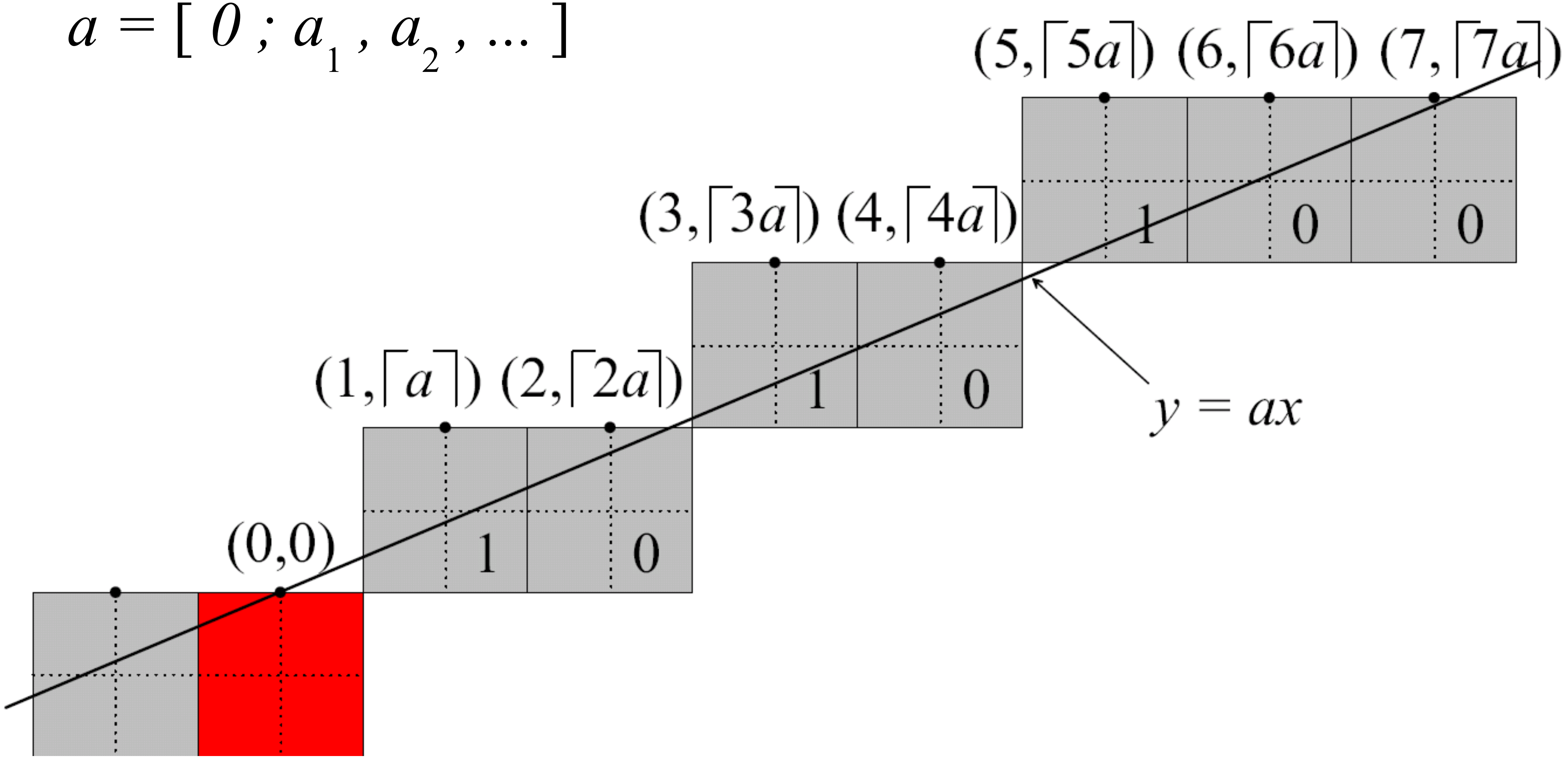


The arithmetical expression of the R'-digitization of the line $y = ax$ for irrational positive a less than 1 :

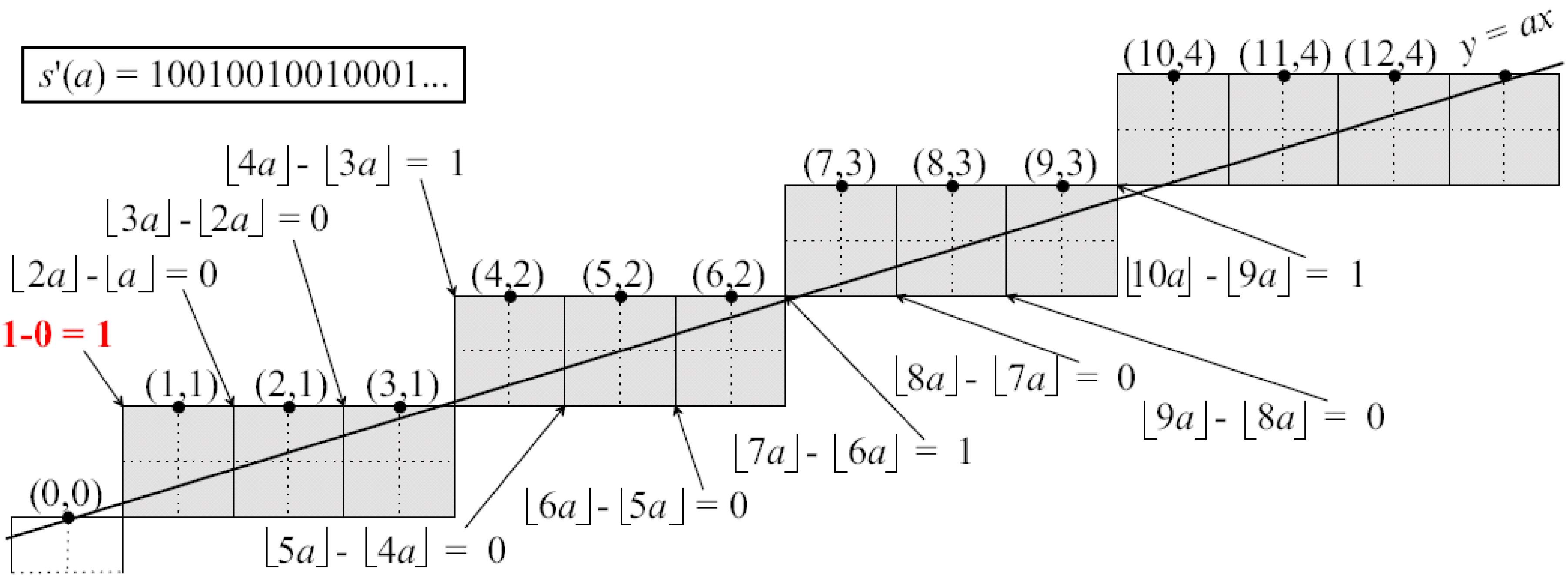
$$D_{R'}(y = ax) = \{(k, \lceil ak \rceil);\ k \in \mathbf{Z}\}$$

The R'-digital line $y = ax$ with irrational slope

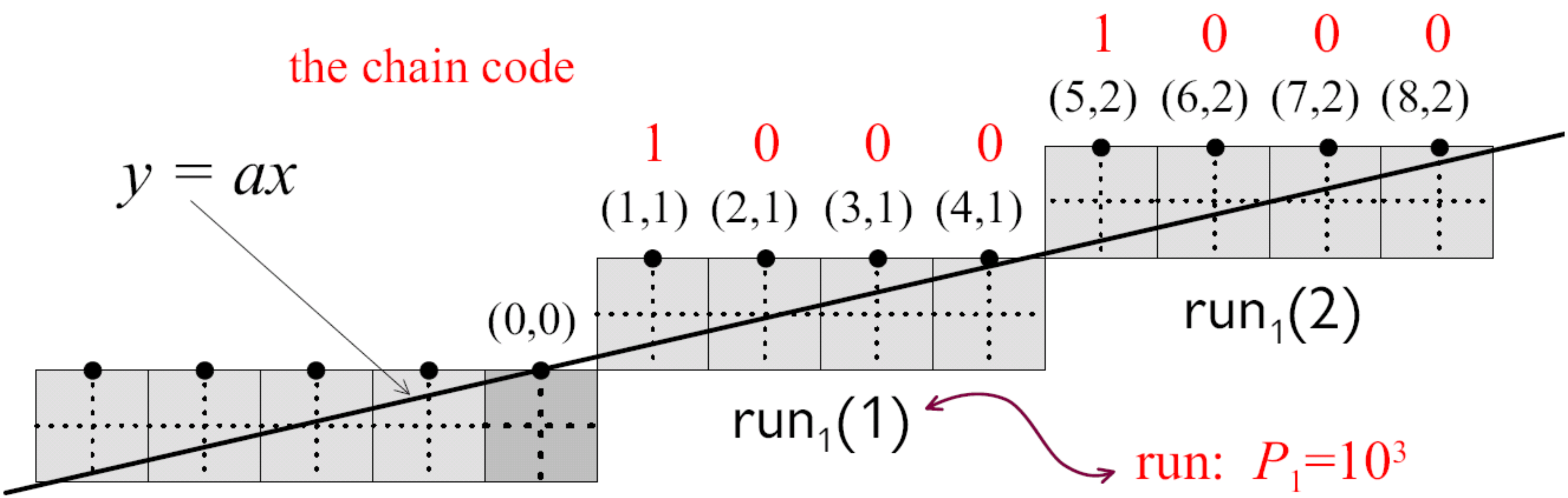
$a = [\, 0\, ;\, a_1\, ,\, a_2\, ,\, \ldots\,]$



The R'-digital line $y = ax$ with slope $a = [\,0\,;\, a_1\,,\, a_2\,,\, \dots\,]$ and the corresponding upper mechanical word $s'(a)$:



Digital geometry – the concept of run



Digital geometry – the concept of run

P_n - the n^{th} prefix according to the run hierarchy

S_n - short run of level n

L_n - long run of level n

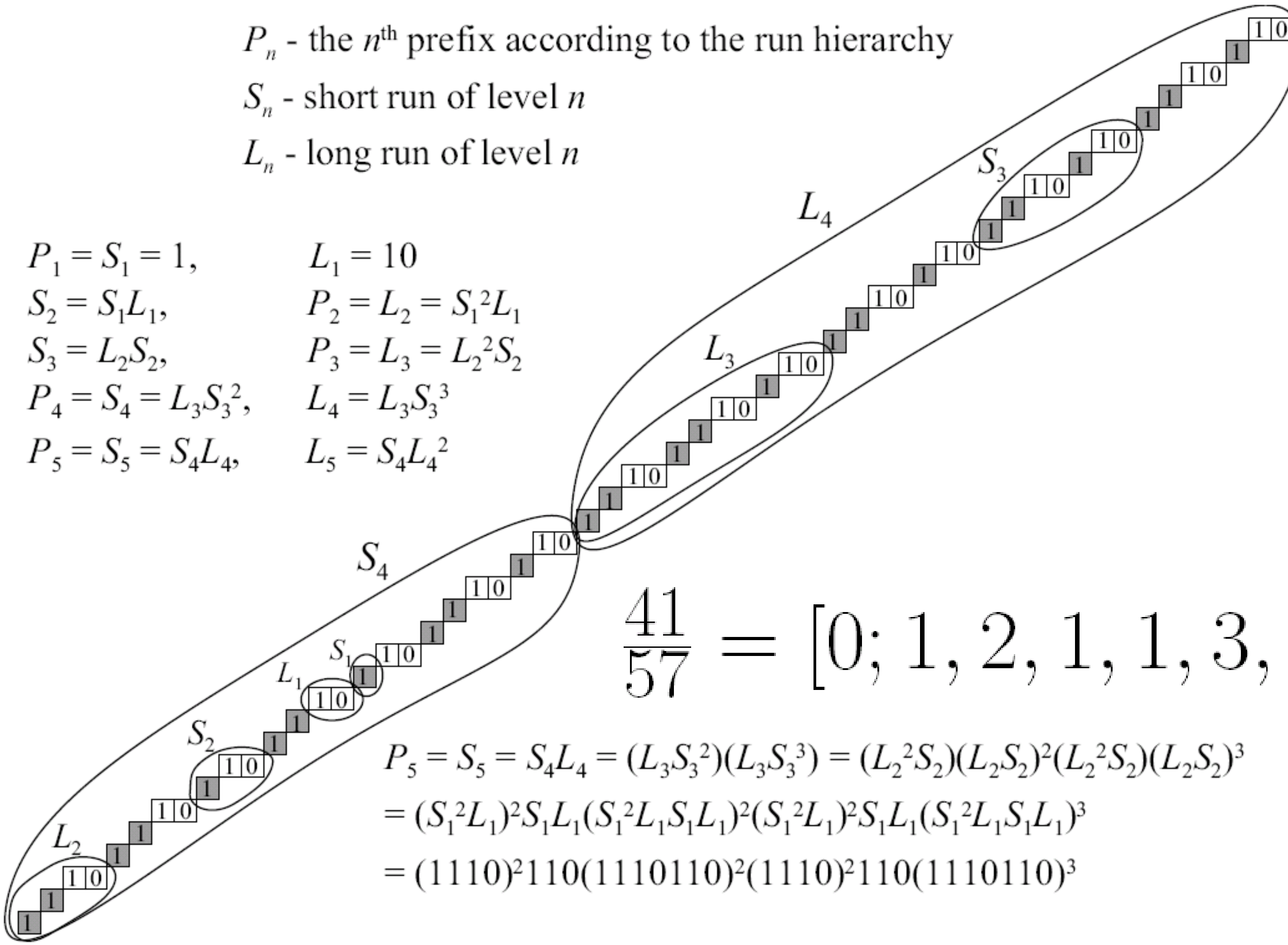
$P_1 = S_1 = 1,$ $\quad$ $L_1 = 10$

$S_2 = S_1 L_1,$ $\quad$ $P_2 = L_2 = S_1^2 L_1$

$S_3 = L_2 S_2,$ $\quad$ $P_3 = L_3 = L_2^2 S_2$

$P_4 = S_4 = L_3 S_3^2,$ $\quad$ $L_4 = L_3 S_3^3$

$P_5 = S_5 = S_4 L_4,$ $\quad$ $L_5 = S_4 L_4^2$



$$\frac{41}{57} = [0; 1, 2, 1, 1, 3, 1, 1]$$

$$P_5 = S_5 = S_4 L_4 = (L_3 S_3^2)(L_3 S_3^3) = (L_2^2 S_2)(L_2 S_2)^2 (L_2^2 S_2)(L_2 S_2)^3$$

$$= (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^2 (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^3$$

$$= (1110)^2 110 (1110110)^2 (1110)^2 110 (1110110)^3$$

Digital geometry – the concept of run

Two run lengths on level 1: runs_1 $S_1=10^m$ and $L_1=10^{m+1}$

S_1

1	0	0	0	0	0	0

L_1

1	0	0	0	0	0	0	0

Two run lengths on level 2: runs_2 $S_2=S_1L_1^{\,k}$ and $L_2=S_1L_1^{\,k+1}$

or $S_2=S_1^{\,k}L_1$ and $L_2=S_1^{\,k+1}L_1$

Two run lengths on level n: runs_n $S_n=S_{n-1}L_{n-1}^{\,l}$ and $L_n=S_{n-1}L_{n-1}^{\,l+1}$

or $S_n=S_{n-1}^{\,l}L_{n-1}$ and $L_n=S_{n-1}^{\,l+1}L_{n-1}$

or $S_n=L_{n-1}S_{n-1}^{\,l}$ and $L_n=L_{n-1}S_{n-1}^{\,l+1}$

or $S_n=L_{n-1}^{\,l}S_{n-1}$ and $L_n=L_{n-1}^{\,l+1}S_{n-1}$

Hierarchy of runs – runs on level $k+1$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

Three questions. About:

the run length on level $k+1$

the main run on level k

the first run on level k

2

Continued fractions

Continued fractions - notation

$$a = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [0; a_1, a_2, a_3, \ldots]$$

Continued fractions – the CF-elements

$$\frac{1}{\frac{1}{a}} = \frac{1}{\left\lfloor \frac{1}{a} \right\rfloor + \mathrm{frac}\left(\frac{1}{a}\right)} = \frac{1}{\left\lfloor \frac{1}{a} \right\rfloor + \frac{1}{\left\lfloor \frac{1}{\mathrm{frac}\left(\frac{1}{a}\right)} \right\rfloor + \mathrm{frac}\left(\frac{1}{\mathrm{frac}\left(\frac{1}{a}\right)}\right)}}$$

a_1

a_2

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Continued fractions – a definition

$$a = [a_0; a_1, a_2, a_3, \ldots]$$

$$\alpha_0 = a; \qquad \text{for} \quad n \geq 0:$$

$$a_n = \lfloor \alpha_n \rfloor, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n}$$

$$= \frac{1}{\text{frac}(\alpha_n)}$$

Continued fractions – an example

$$\frac{13}{41} = \frac{1}{\frac{41}{13}} = \frac{1}{3+\frac{2}{13}} = \frac{1}{3+\frac{1}{\frac{13}{2}}} =$$

$$\frac{1}{3+\frac{1}{6+\frac{1}{2}}} = [0;3,6,2].$$

Continued fractions and decimal expansions

$$0.1111\cdots = \frac{1}{9} = [0;9]$$

$$[0;1,1,1,\ldots] = \frac{\sqrt{5}-1}{2} = 0.6180339887\ldots$$

Periodicity of continued fractions

The CF-expansion of a is periodic

a is a quadratic surd

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Quadratic surd (quadratic irrational) ...

... is an algebraic number of the second degree, i.e.:

is irrational and is a root of some equation

$$a_2 x^2 + a_1 x + a_0 = 0$$

with integer coefficients.

$\frac{\sqrt{5}-1}{2}$ is a root of $x^2 + x - 1 = 0$

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Periodicity of continued fractions

$$[0; 1, 1, 1, 1, \ldots] = \,?$$

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Periodicity of continued fractions

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x

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Periodicity of continued fractions

$$[0; 1, 1, 1, 1, \ldots] = ?$$

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{1}{1+x}$$

x

Periodicity of continued fractions

$$[0;1,1,1,1,\ldots] = ?$$

$$x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{1}{1+x}$$

$$x = \frac{1}{1+x} \quad \Leftrightarrow \quad x^2 + x - 1 = 0$$

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Periodicity of continued fractions

$$[0; 1, 1, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2}$$

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Periodicity of continued fractions

The CF-expansion of a is periodic

$$\Downarrow$$

Euler 1737

a is a quadratic surd

Periodicity of continued fractions

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The CF-expansion of a is periodic

Lagrange 1770

Euler 1737

a is a quadratic surd

Continued fractions and decimal expansions

		CF-expansion	decimal expansion
finite		rational	rational
infinite	periodic	irrational (quadratic surd)	rational
	aperiodic	irrational (no quadratic surd)	irrational

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Continued fractions – periodic patterns

(Euler 1737)

$$e-2 = [0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots]$$

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Continued fractions – periodic patterns

(Euler 1737)

$$e-2 = [0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots]$$

$$= [0; \overline{1, 2k, 1}]_{k=1}^{\infty}$$

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Continued fractions – periodic patterns

(Euler 1737)

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$$= [0; \overline{1, 2k, 1}]_{k=1}^{\infty}$$

for $k \geq 2$

$$\sqrt[n]{e} - 1 = [0; \overline{(2k-1)n - 1, 1, 1}]_{k=1}^{\infty}$$

Continued fractions – periodic patterns

(Lambert 1761)

for $k \geq 2$

$$\tan(1/k) = [0; k-1, \overline{1, (2n+1)k-2}]_{n=1}^{\infty}$$

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Continued fractions – periodic patterns

(Lambert 1761)

for $k \geq 2$

$$\tan(1/k) = [0; k-1, \overline{1, (2n+1)k-2}]_{n=1}^{\infty}$$

$$\tan(1/2) = [0; 1, 1, 4, 1, 8, 1, 12, 1, 16, \ldots]$$

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Combinatorics on CFs

Important issues

Two equivalence relations on the set of slopes

A new fixed point theorem for words

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Important issues

Two equivalence relations on the set of slopes

A new fixed point theorem for words

A new CF-description (essential 1's, run hierarchy)

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Two equivalence relations on the set of slopes

A new CF-description (essential 1's, run hierarchy)

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An informal introduction to the equivalence relations on CFs

$$a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$$

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$$a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, \quad b_3, \quad b_4, \quad b_5, \quad b_6, b_7, \quad b_8, \quad b_9, \quad b_{10}, \quad b_{11}, \quad b_{12}, \ldots)$$

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$$(b_k)_{k=1}^{\infty} = (1,$$

$$(b_k)_{k=1}^{\infty} = (b_1,$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2,$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2,$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 + 1$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, \; b_3,$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8,$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6,$$

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$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7,$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11},$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1,$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10},$$

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$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1, 1 \oplus a_{16},$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11},$$

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An informal introduction to the equivalence relations on CFs

$$a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$$

$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1, 1 \oplus a_{16}, a_{17}, \ldots)$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots)$$

$$a' = [0; 1, 1, a_2 - 1, 2, a_5, 1, 1, 1, a_8 - 1, a_9, a_{11} + 1, 1, a_{12} - 1, 2, 1, a_{16}, a_{17}, \ldots]$$

An informal introduction to the equivalence relations on CFs

$$a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$$

$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1, 1 \oplus a_{16}, a_{17}, \ldots)$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots)$$

$$a' = [0; 1, 1, a_2 - 1, 2, a_5, 1, 1, 1, a_8 - 1, a_9, a_{11} + 1, 1, a_{12} - 1, 2, 1, a_{16}, a_{17}, \ldots]$$

An informal introduction to the equivalence relations on CFs

$$a_3 \quad a_6 \quad a_{10} \quad a_{13} \quad a_{15}$$

$$a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$$

$$(b_k)_{k=1}^{\infty} = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1, 1 \oplus a_{16}, a_{17}, \ldots)$$

$$(b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots)$$

$$(s_k)_{k \in I} = (3, 6, 10, 13, 15, \ldots)$$

The index jump function

$$a = [0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots]$$

$$i_a : \mathbf{N}^+ \to \mathbf{N}^+$$

$$i_a(1) = 1,\ i_a(2) = 2, \text{ for } n \geq 2:$$

$$i_a(n+1) = i_a(n) + 1 + \delta_1(a_{i_a(n)})$$

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The index jump function: an example

$$a = [0;\ \overset{b_1}{1},\ \overset{b_2}{a_2},\ \overbrace{\underline{1},\ 1}^{b_3},\ \overset{b_4}{a_5},\ \overbrace{\underline{1},\ 1}^{b_5},\ \overset{b_6}{a_8},\ \overset{b_7}{a_9},\ \overbrace{\underline{1},\ a_{11}}^{b_8},\ \overset{b_9}{a_{12}},\ \overbrace{\underline{1},\ 1}^{b_{10}},\ \overbrace{\underline{1},\ a_{16}}^{b_{11}},\ \overset{b_{12}}{a_{17}},\ \ldots]$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \ldots$$

$$(i_a(k))_{k \in \mathbf{N}^+} = (\ 1,\ 2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17,\ \ldots)$$

An essential 1 is a CF-element equal to 1
and indexed by a value of the index jump function.

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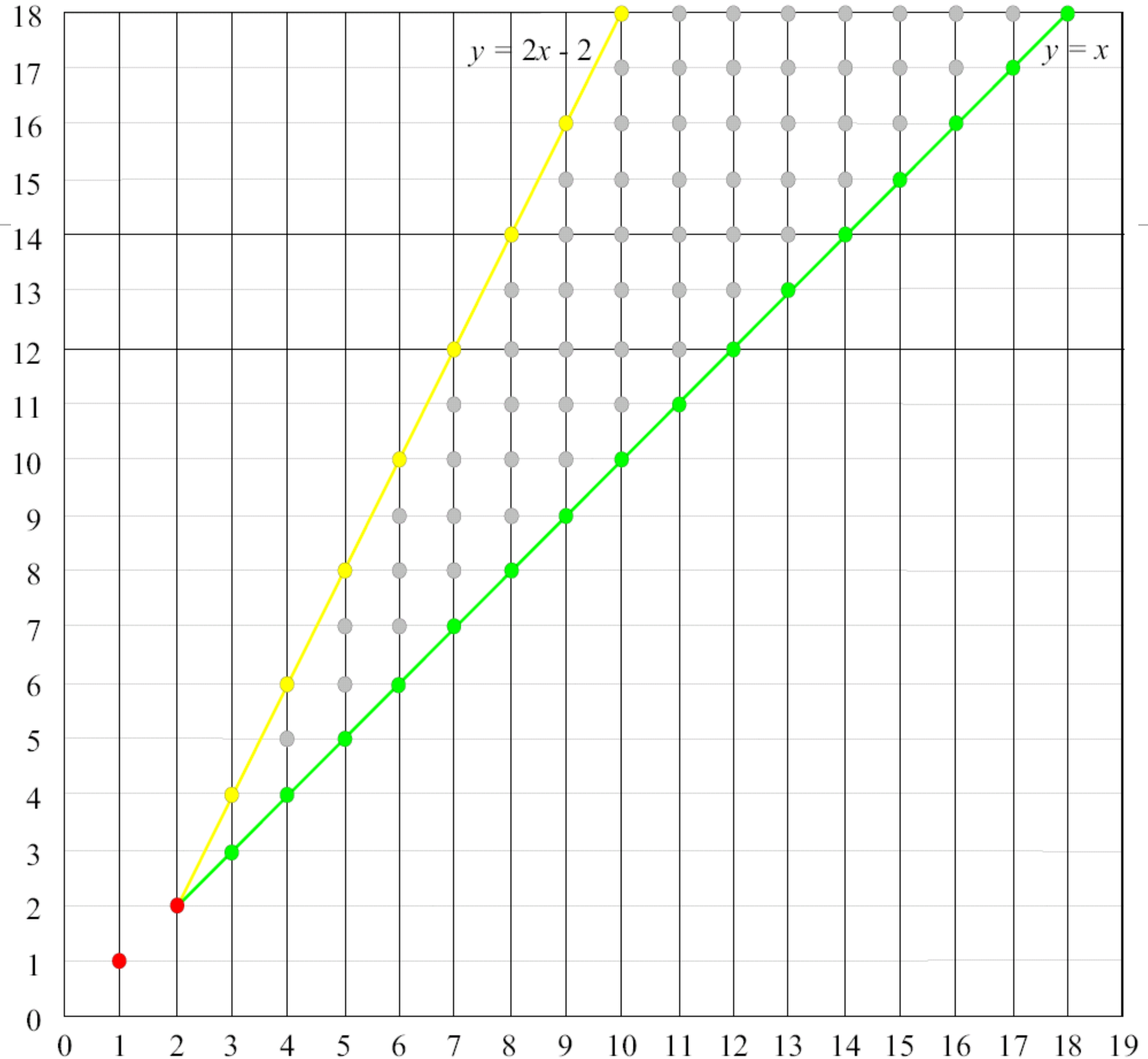
The index jump function - properties

Its values are positive integers

The function is increasing

For each slope a and for each positive integer n

$$i_a(n+1) - i_a(n) = 1 \quad \text{or} \quad i_a(n+1) - i_a(n) = 2$$


y = 2x - 2
y = x
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

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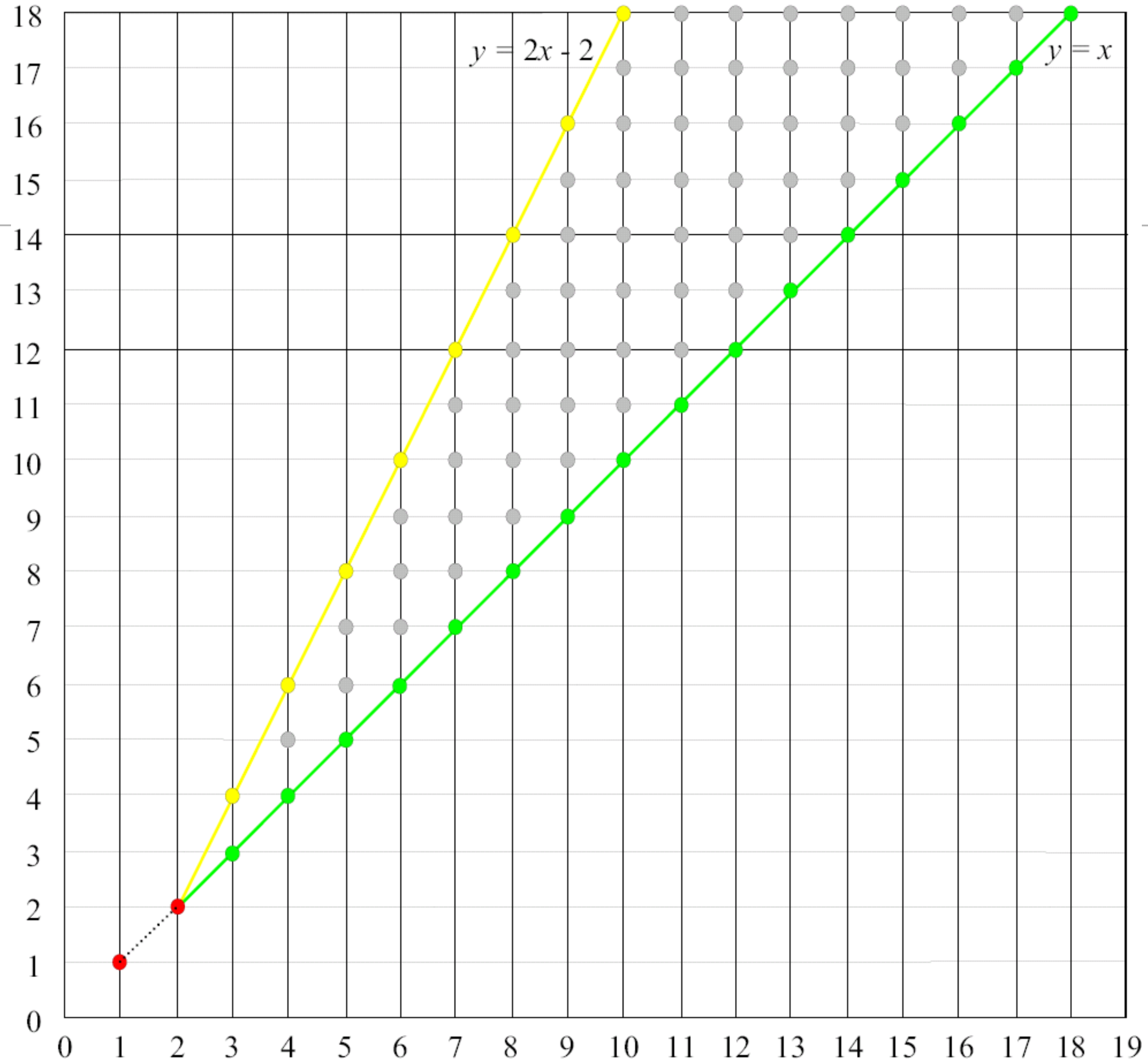

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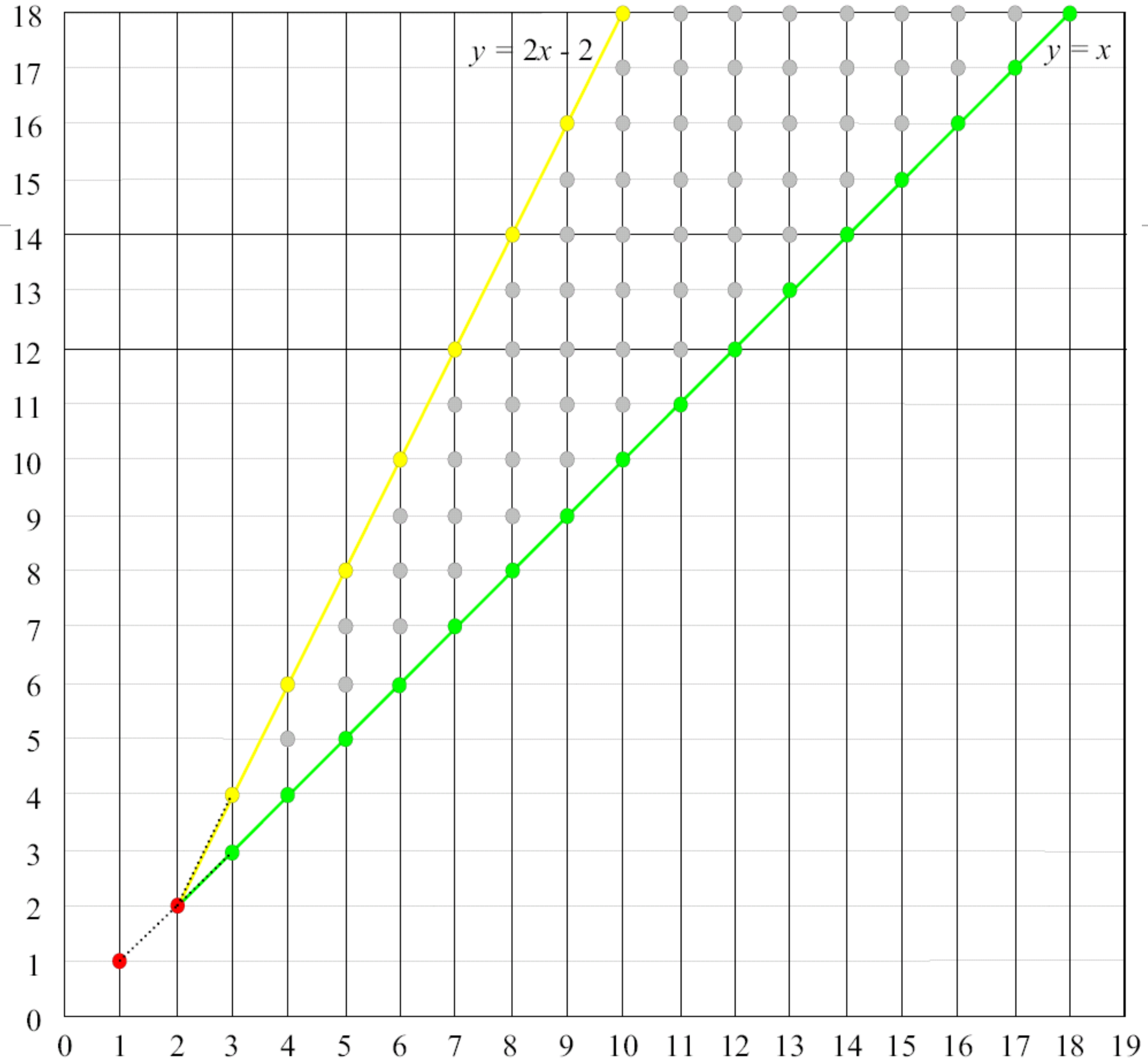

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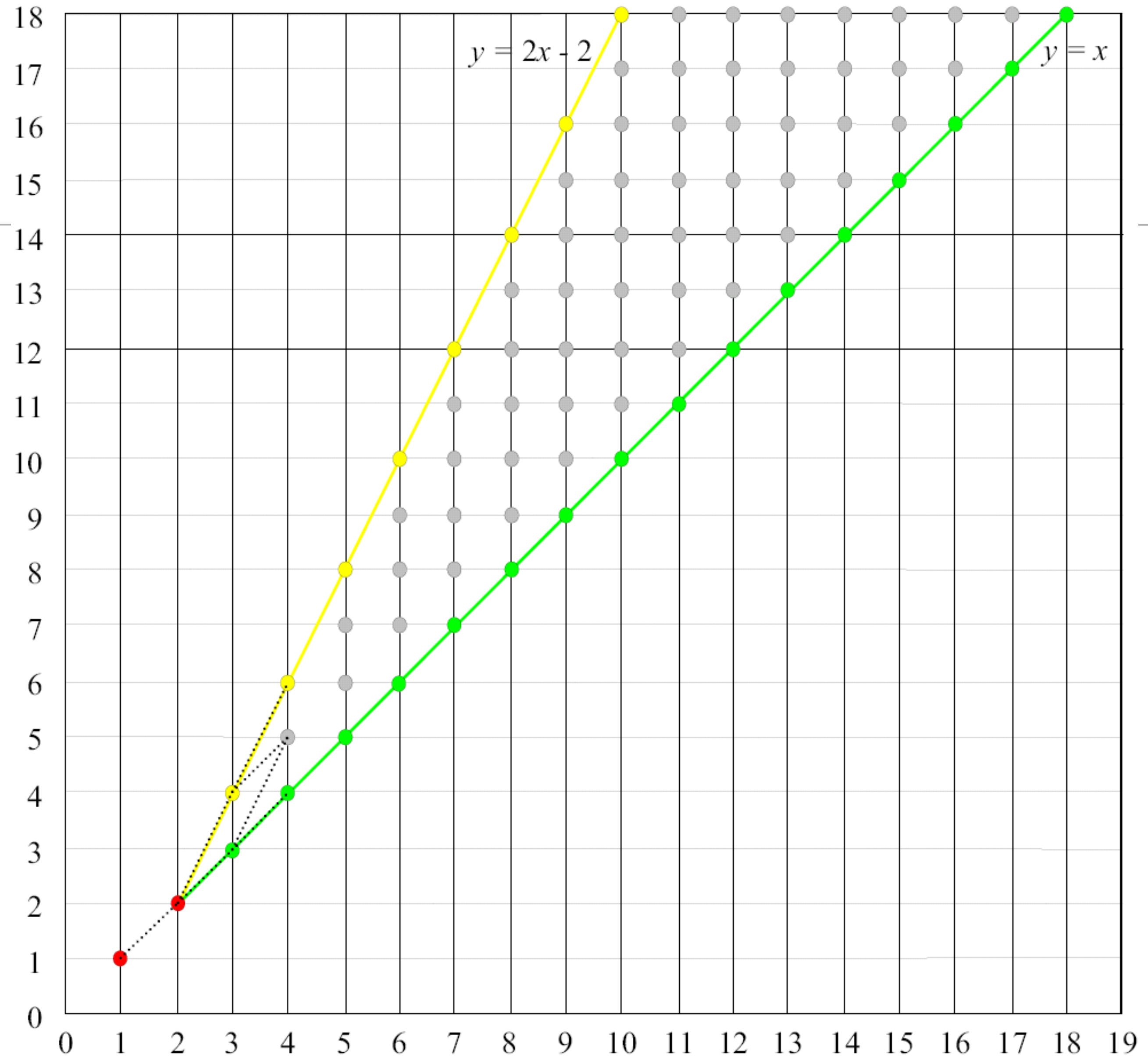

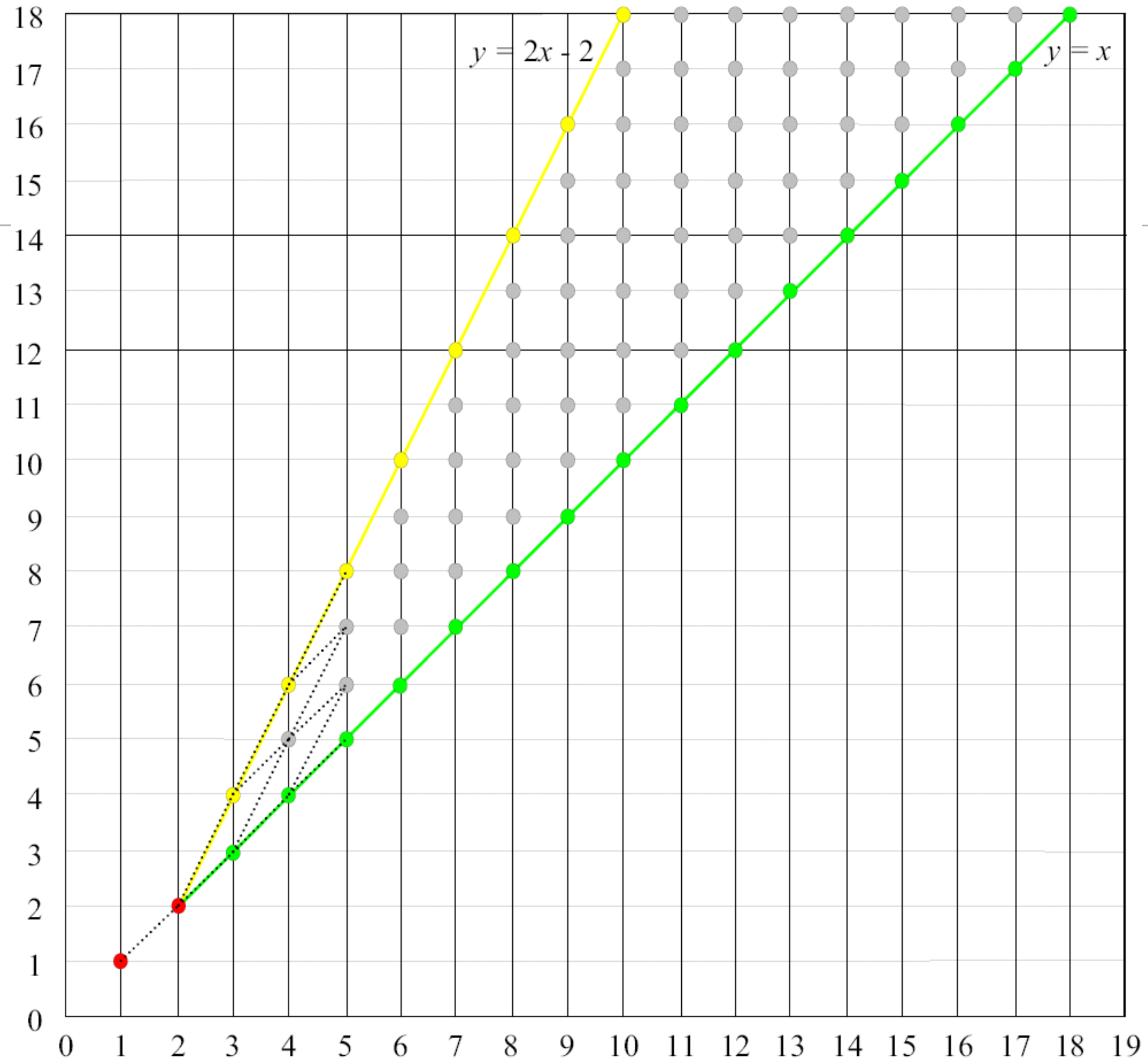
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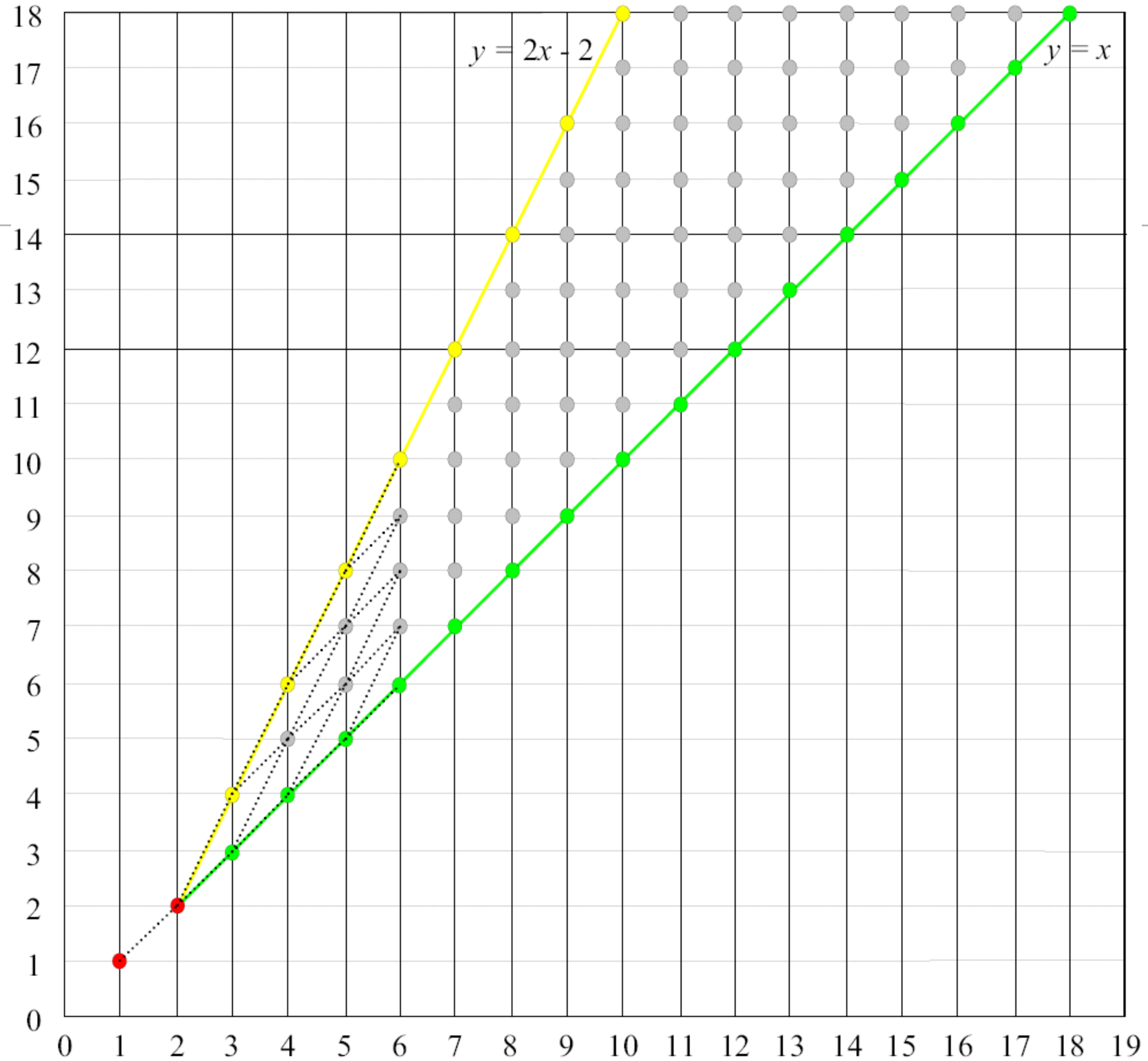

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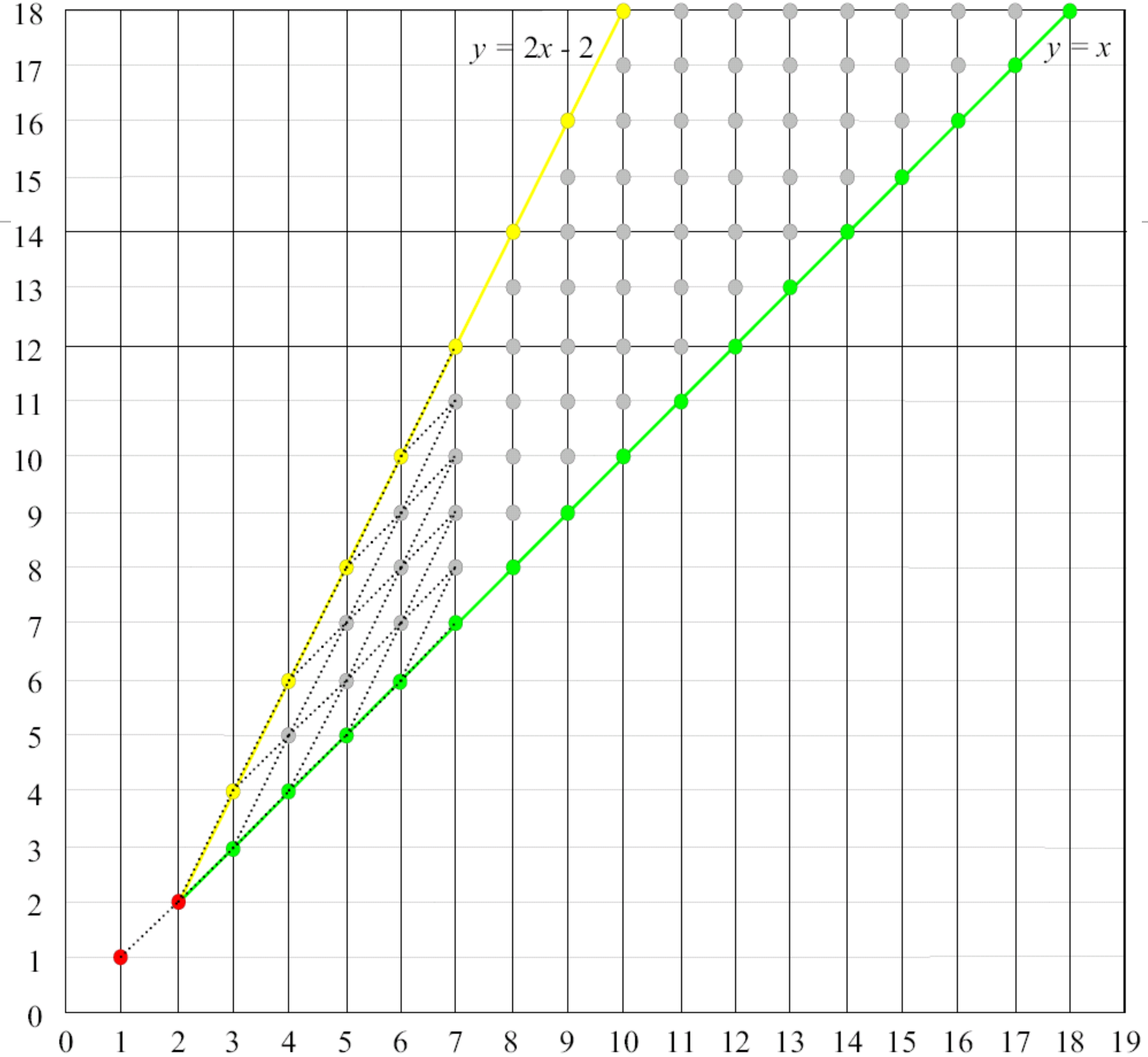

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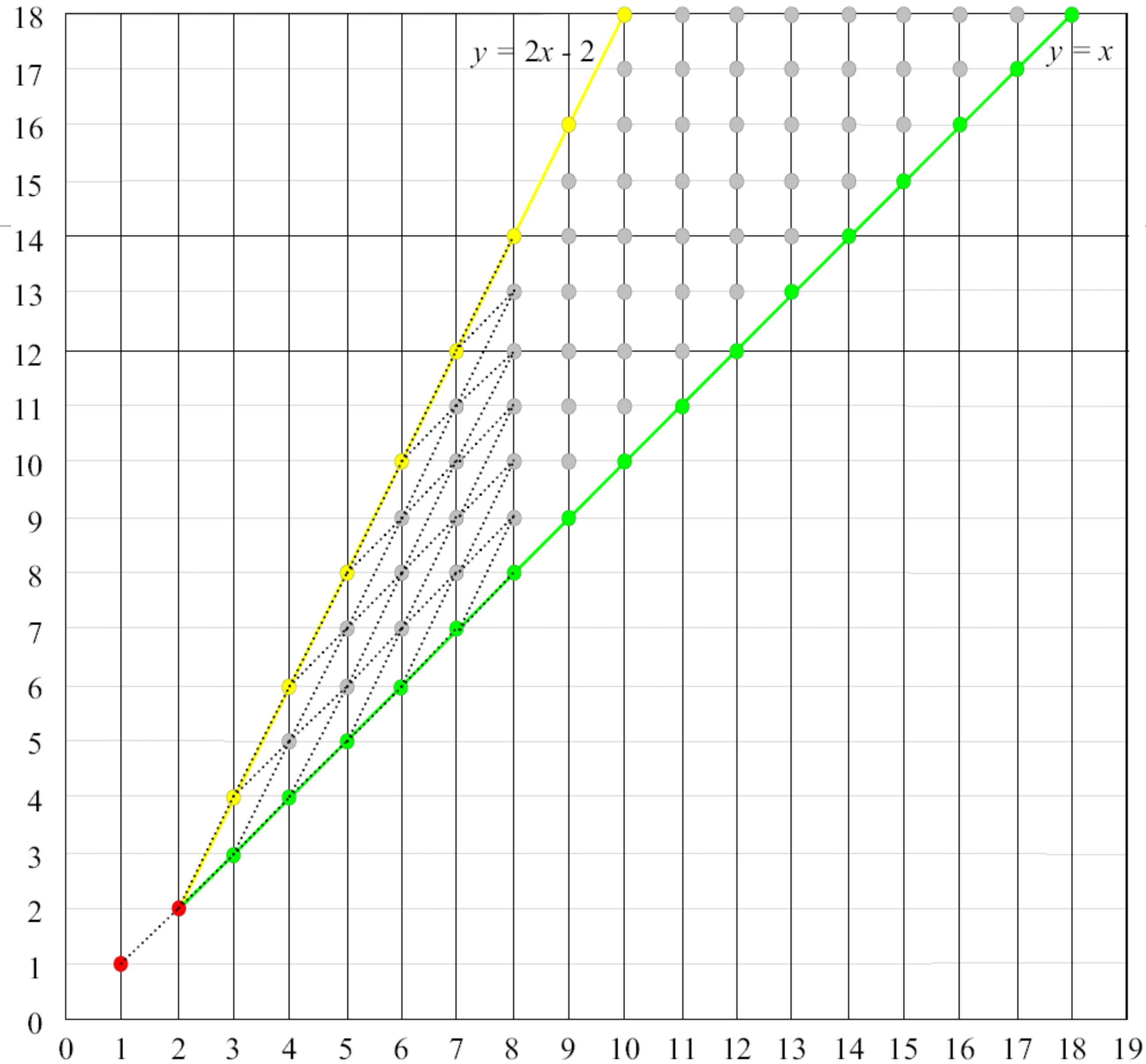
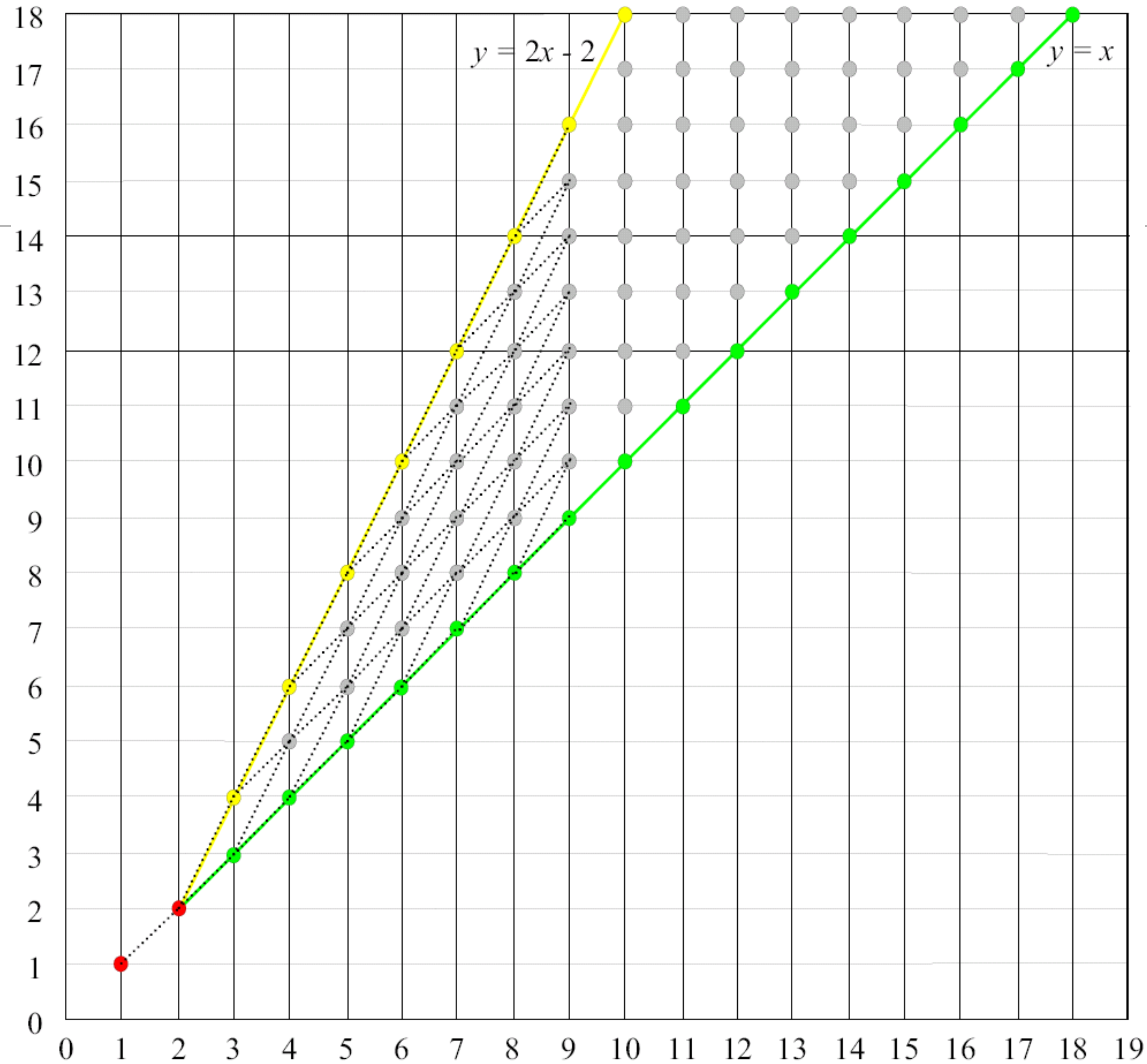
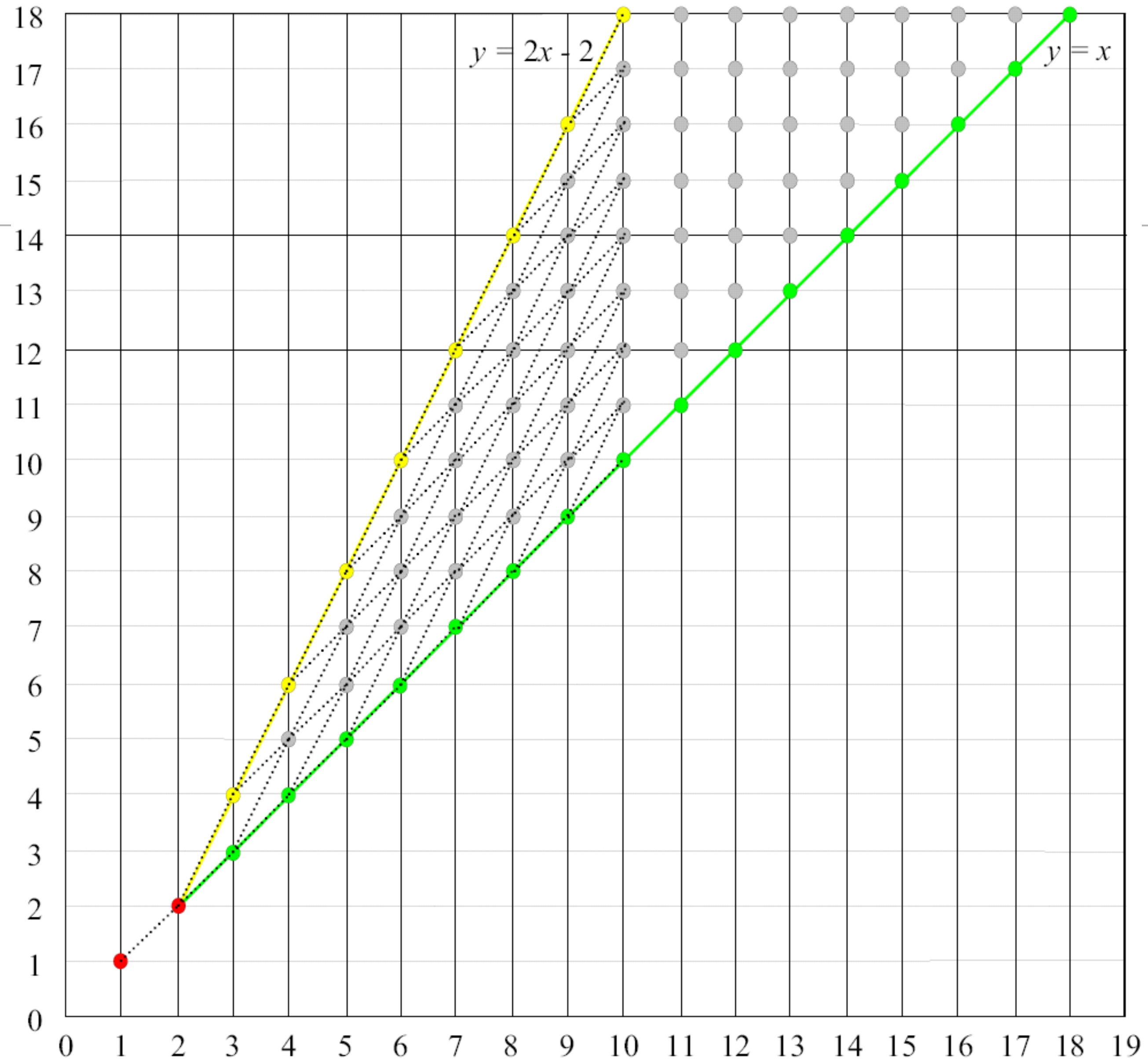
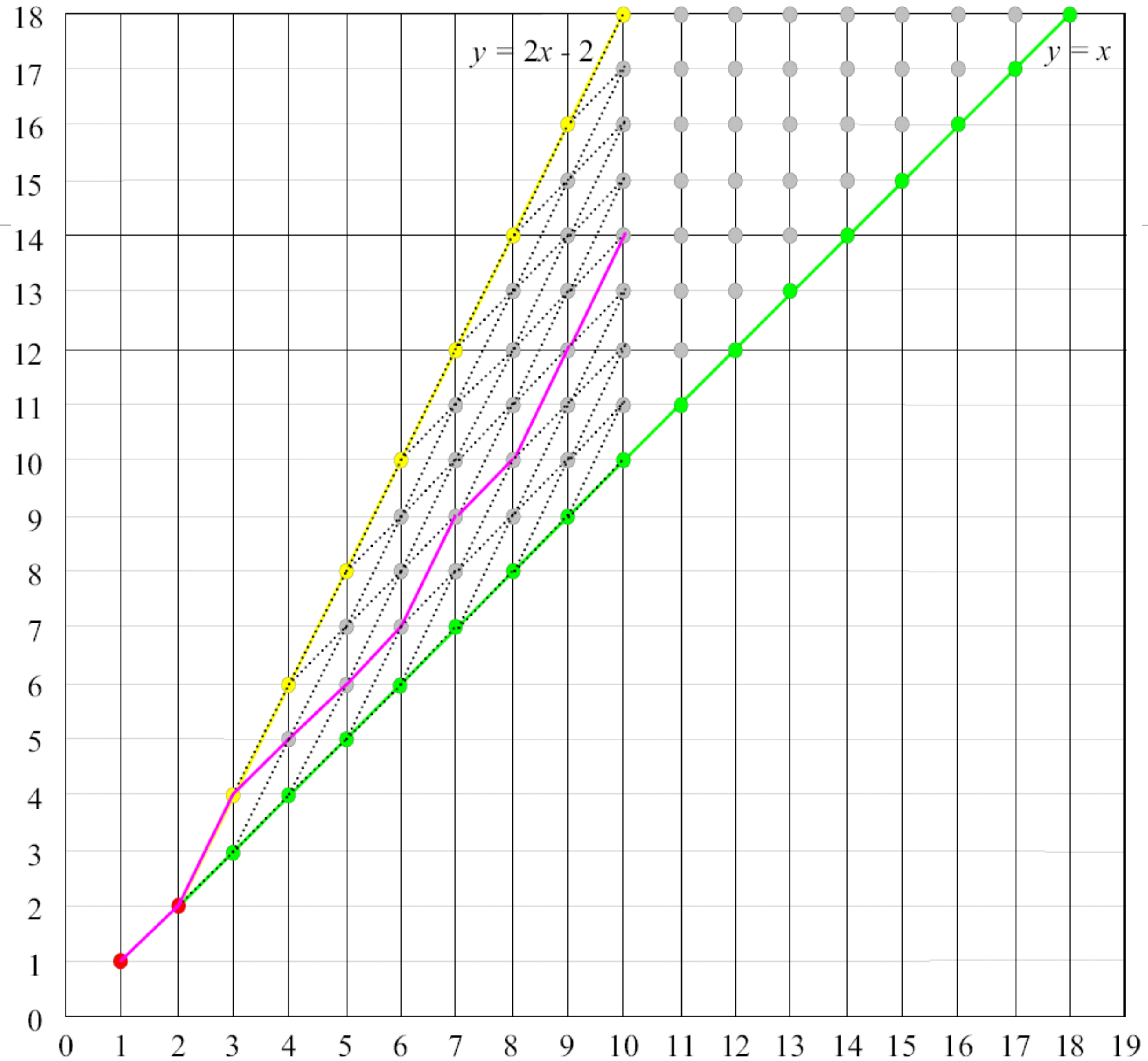




y = 2x - 2
y = x
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19


y = 2x - 2
y = x
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19


y = 2x - 2
y = x
0
1
2
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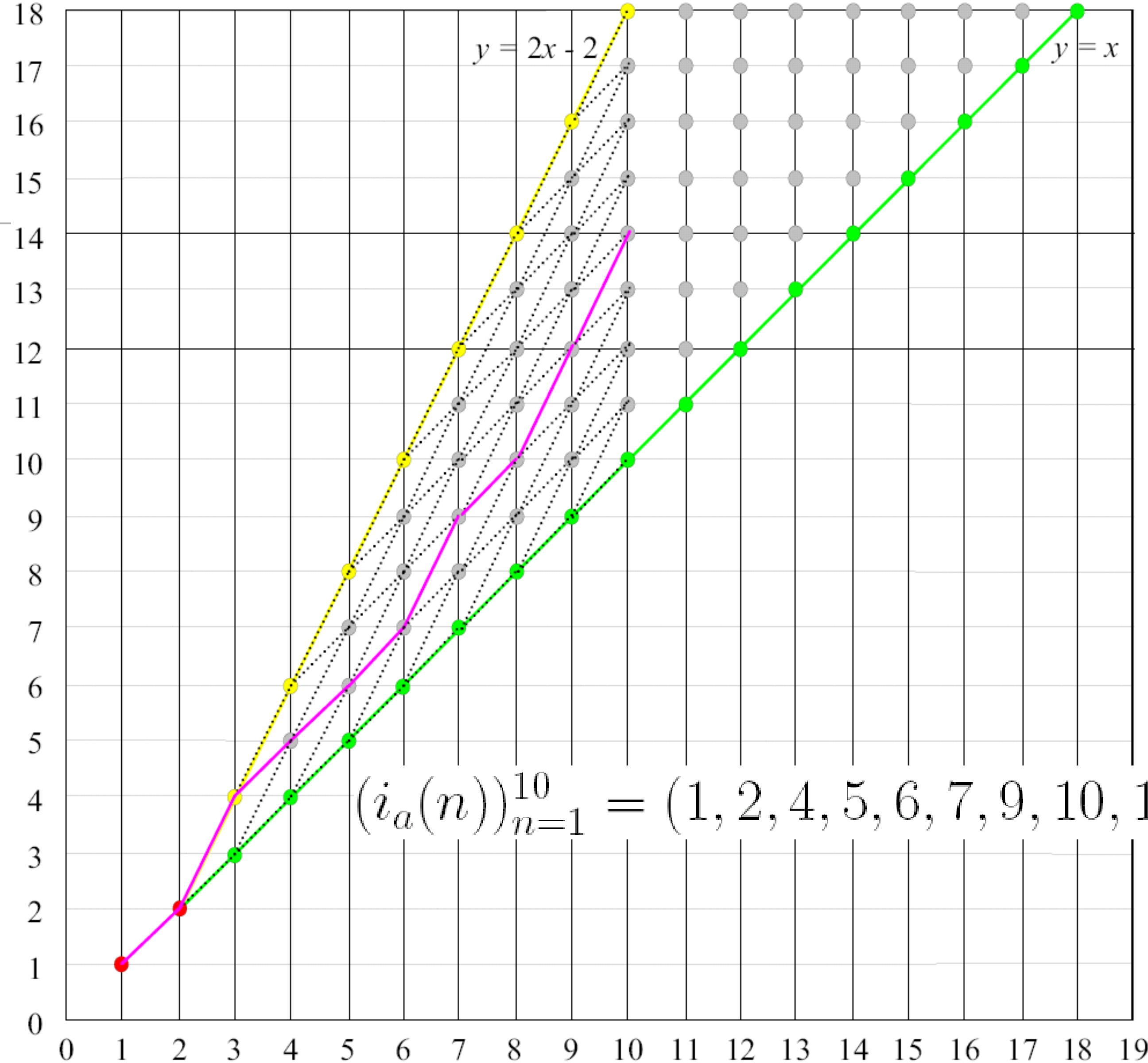

$$(i_a(n))_{n=1}^{10} = (1, 2, 4, 5, 6, 7, 9, 10, 12, 14)$$

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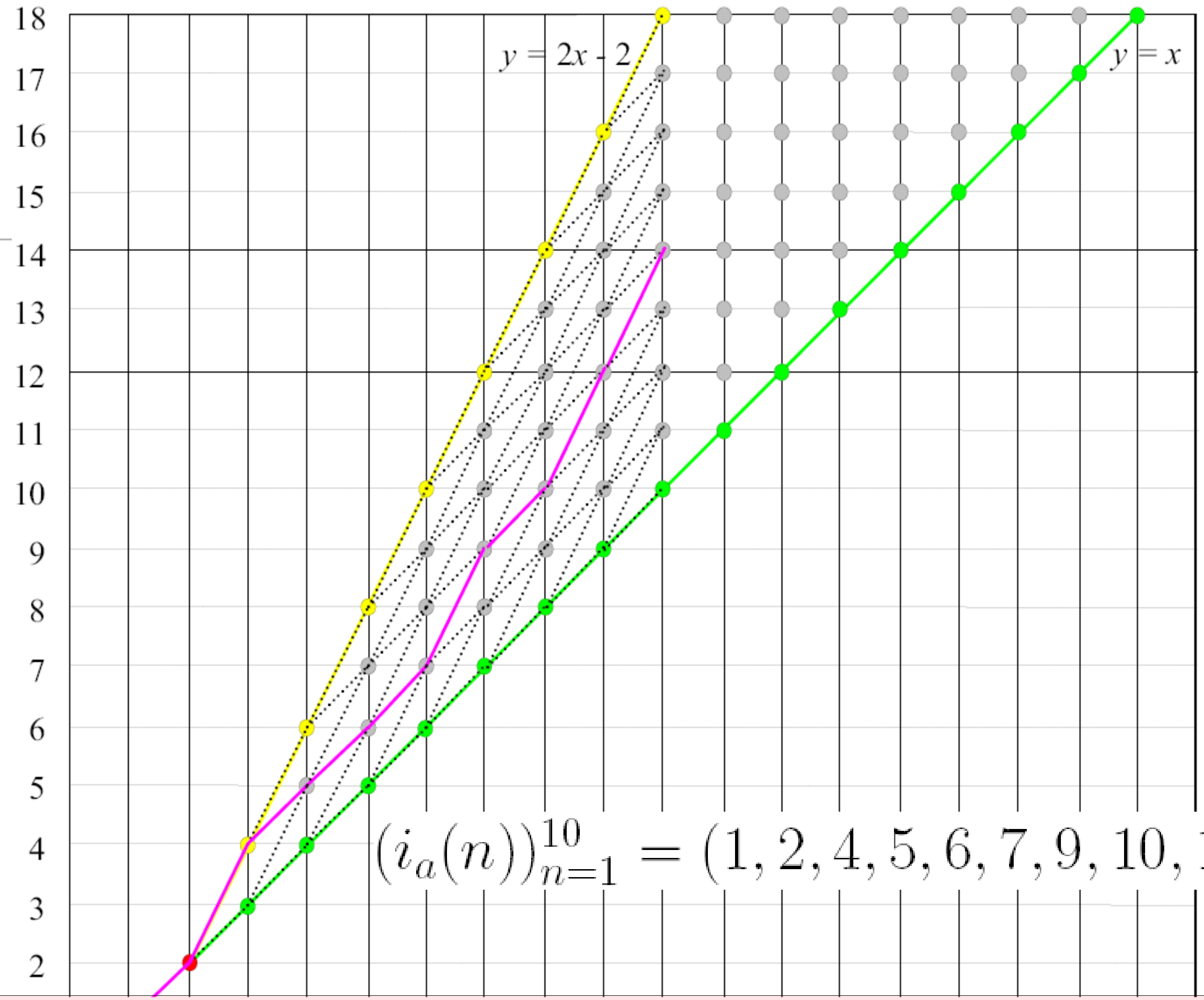

$$(i_a(n))_{n=1}^{10} = (1, 2, 4, 5, 6, 7, 9, 10, 12, 14)$$

$$a = [0; a_1, 1, a_3, a_4, a_5, a_6, 1, a_8, a_9, 1, a_{11}, 1, a_{13}, a_{14}, \ldots]$$

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How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$$a_{i_a(k+1)}$$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

$$i_a(k+1)$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

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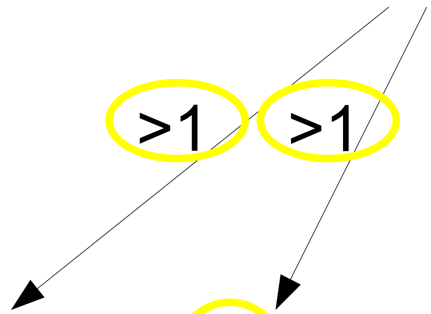
$$a_{i_a(k+1)}$$



$L_k S_k^m$ $\quad$ $S_k^m L_k$ $\quad$ $L_k^m S_k$ $\quad$ $S_k L_k^m$

$$i_a(k+1)$$

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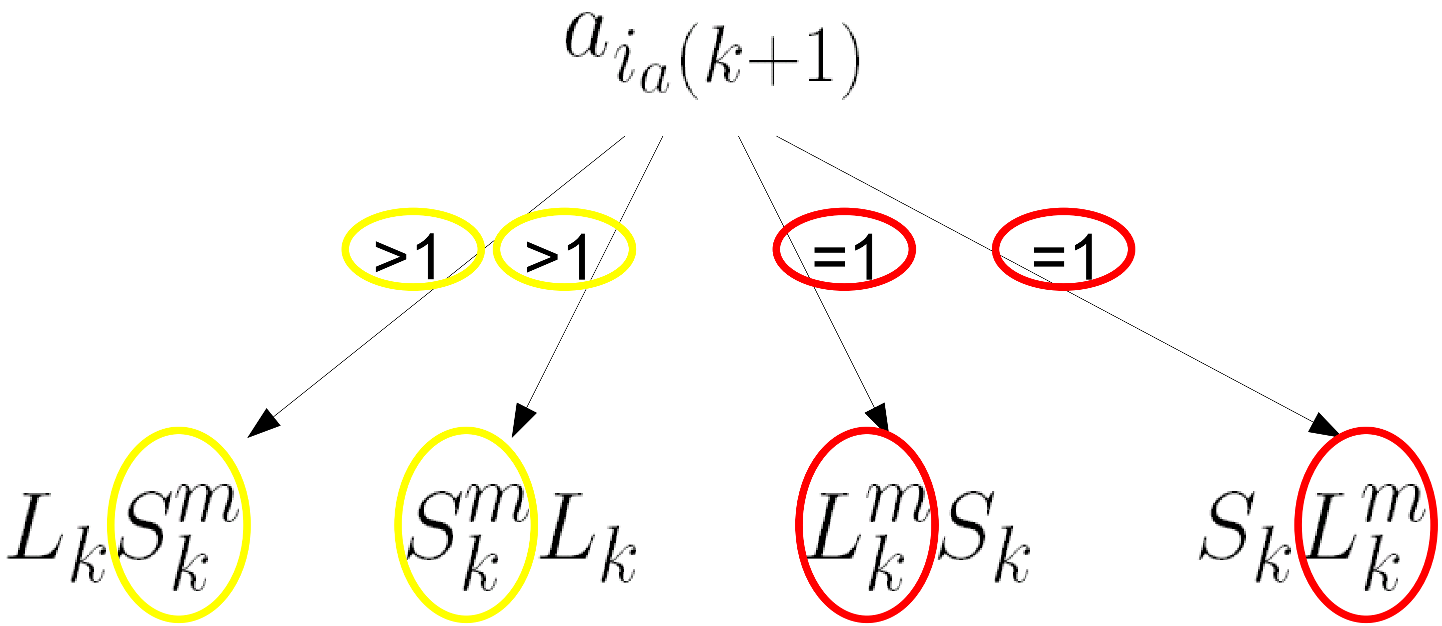
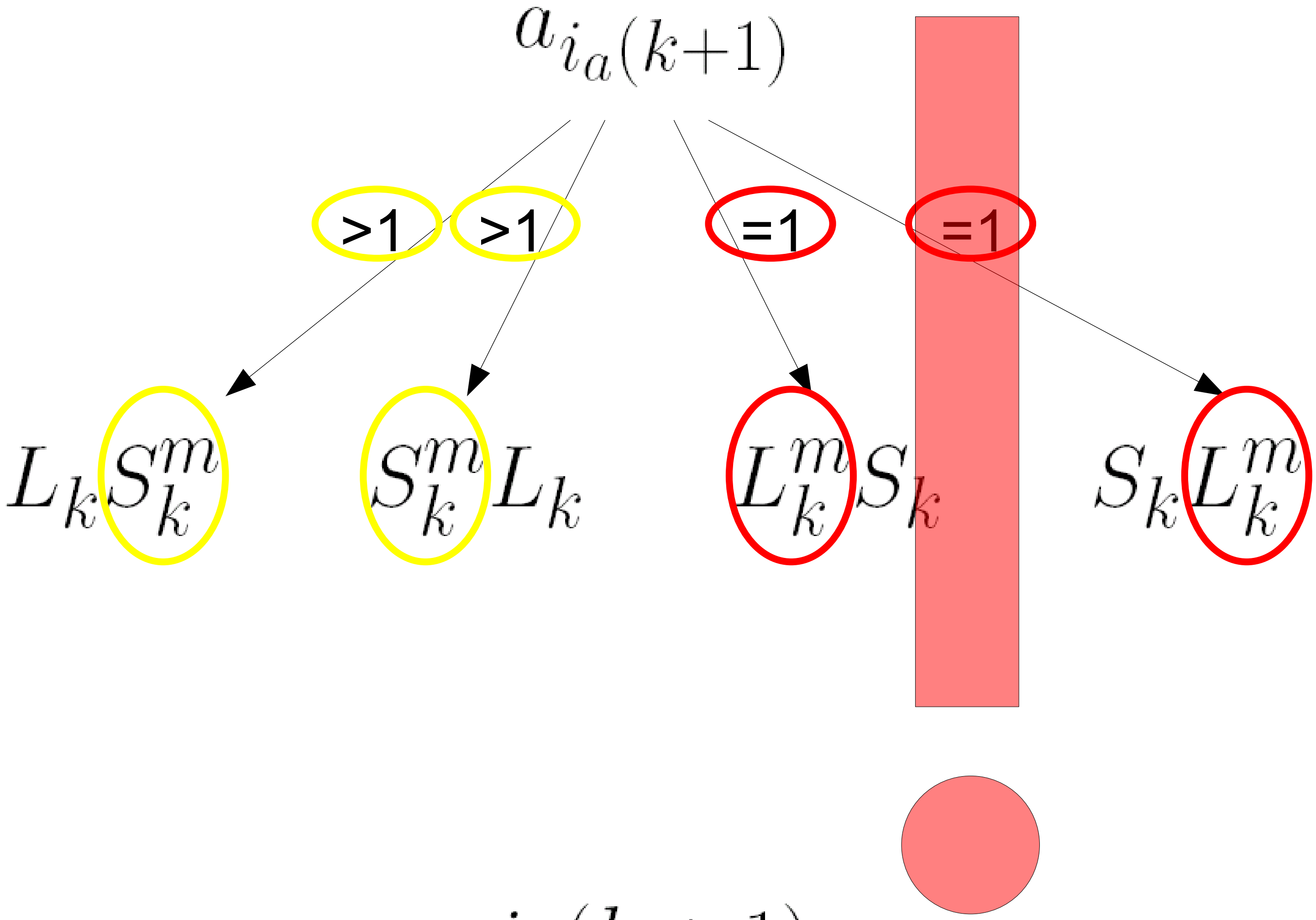
How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}



$$i_a(k+1)$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}



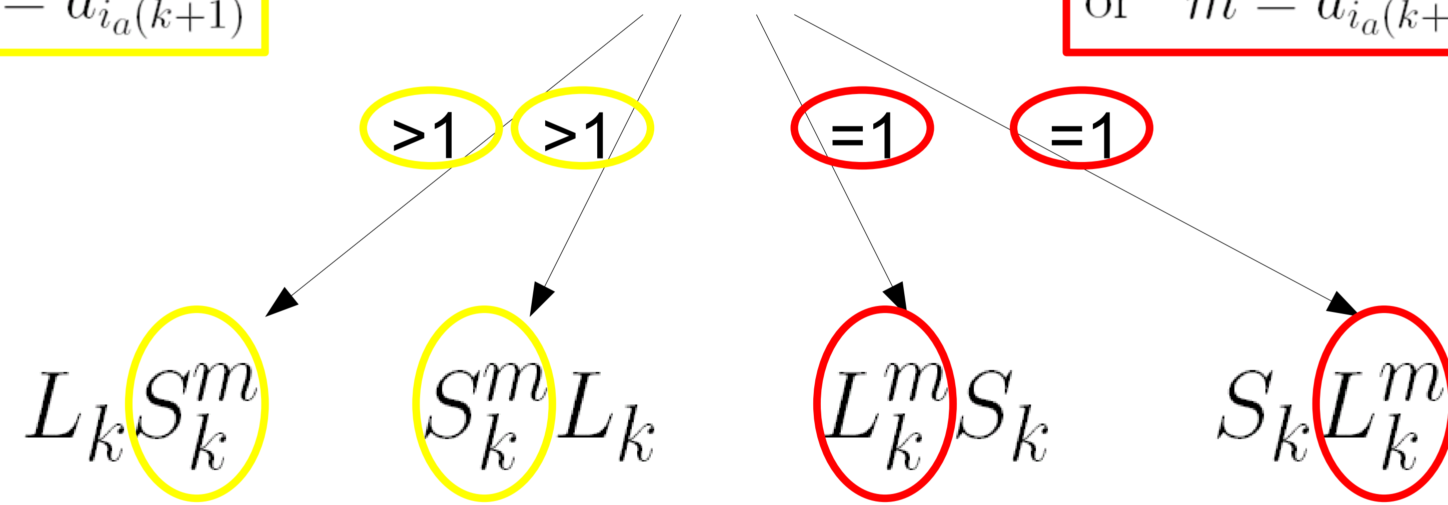
$i_a(k+1)$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$m = a_{i_a(k+1)} - 1$
or $m = a_{i_a(k+1)}$

$a_{i_a(k+1)}$

$m = a_{i_a(k+1)+1}$
or $m = a_{i_a(k+1)+1} + 1$



$i_a(k+1)$

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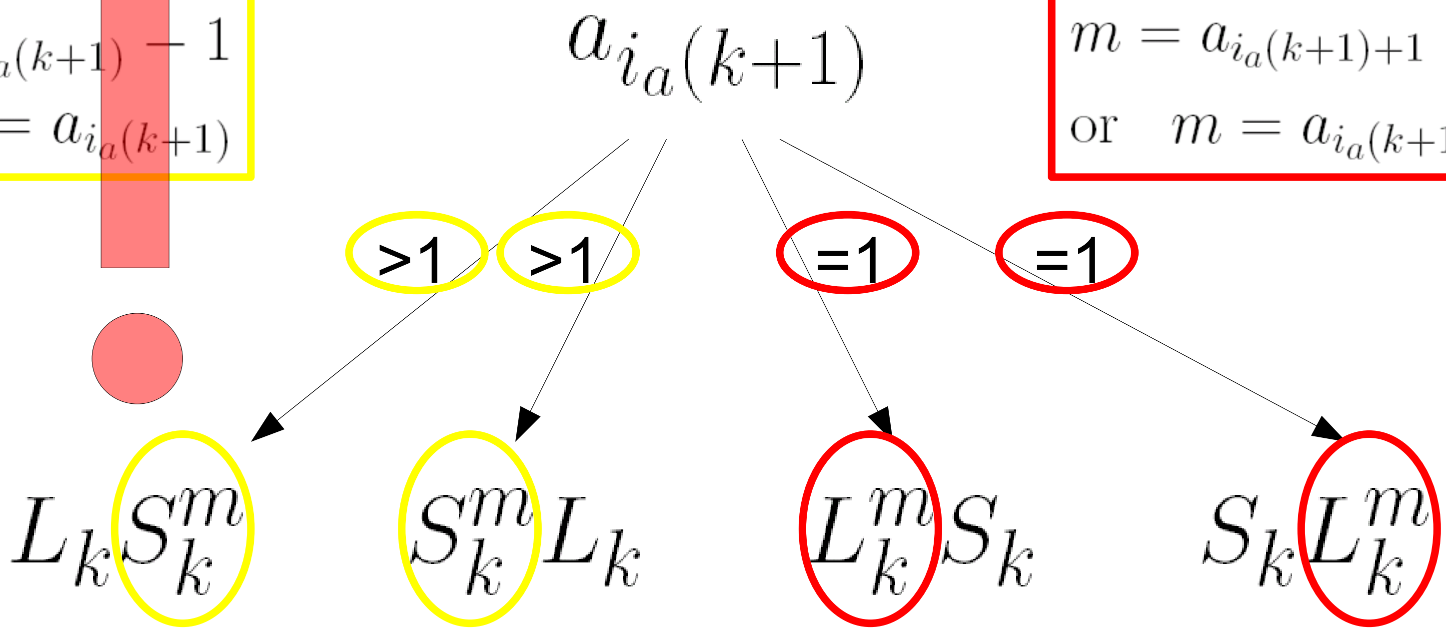
How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$m = a_{i_a(k+1)} - 1$

or $\quad m = a_{i_a(k+1)}$

$m = a_{i_a(k+1)+1}$

or $\quad m = a_{i_a(k+1)+1} + 1$



$i_a(k+1)$

The sequence of length specification for a

$$b_1 = a_1 \quad \text{and, for} \quad n \geq 2:$$

$$b_n = \begin{cases} a_{i_a(n)}, & a_{i_a(n)} \neq 1 \\ 1 + a_{i_a(n)+1}, & a_{i_a(n)} = 1 \end{cases}$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$$a_{i_a(k+1)}$$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

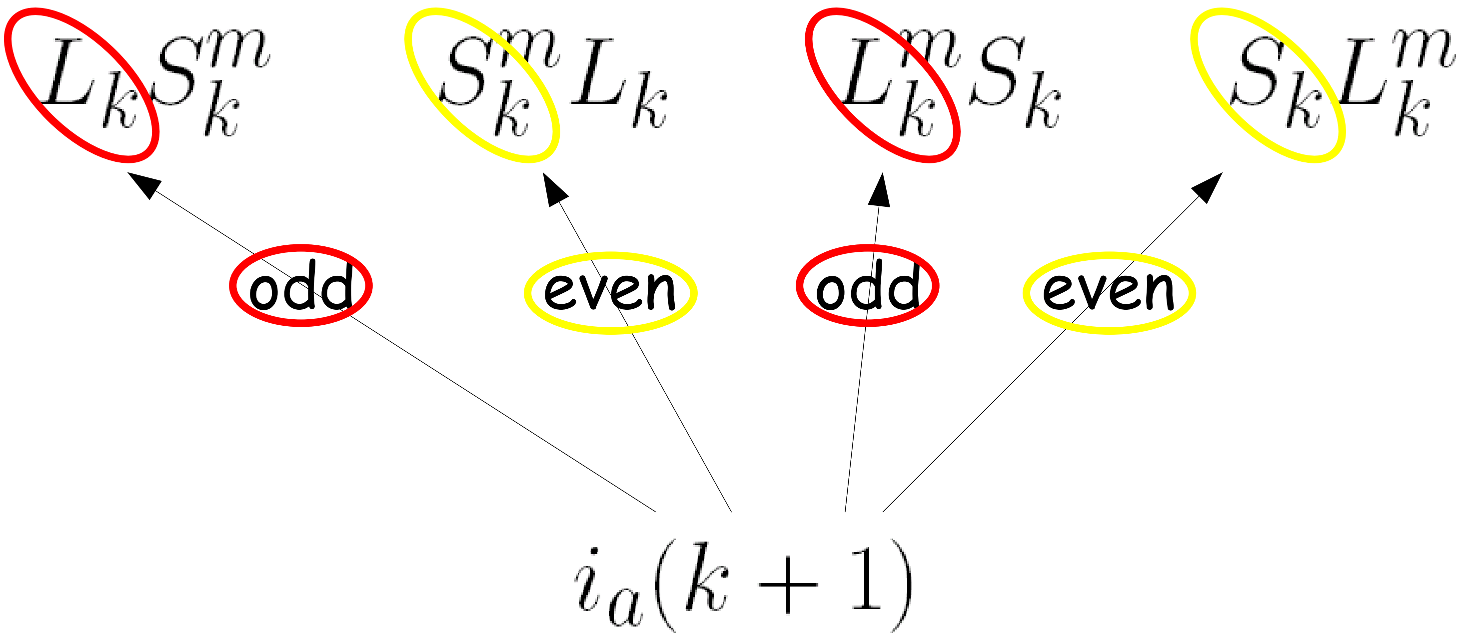
odd

odd

$$i_a(k+1)$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$$a_{i_a(k+1)}$$



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The index jump function: how it describes the runs

$$a = [0;\ \overset{b_1}{1},\ \overset{b_2}{a_2},\ \overbrace{\underline{1},\ 1}^{b_3},\ \overset{b_4}{a_5},\ \overbrace{\underline{1},\ 1}^{b_5},\ \overset{b_6}{a_8},\ \overset{b_7}{a_9},\ \overbrace{\underline{1},\ a_{11}}^{b_8},\ \overset{b_9}{a_{12}},\ \overbrace{\underline{1},\ 1}^{b_{10}},\ \overbrace{\underline{1},\ a_{16}}^{b_{11}},\ \overset{b_{12}}{a_{17}},\ \ldots]$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \ldots$$

$$(i_a(k))_{k \in \mathbf{N}^+} = (\ 1,\ 2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17,\ \ldots)$$

Essential 1's are extremely important in description of runs.

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Digitization levels

$$a = [0;\ \overset{b_1}{1},\ \overset{b_2}{a_2},\ \overbrace{\underline{1},\ 1}^{b_3},\ \overset{b_4}{a_5},\ \overbrace{\underline{1},\ 1}^{b_5},\ \overset{b_6}{a_8},\ \overset{b_7}{a_9},\ \overbrace{\underline{1},\ a_{11}}^{b_8},\ \overset{b_9}{a_{12}},\ \overbrace{\underline{1},\ 1}^{b_{10}},\ \overbrace{\underline{1},\ a_{16}}^{b_{11}},\ \overset{b_{12}}{a_{17}},\ \ldots]$$

Level :	1	2	3	4	5	6	7	8	9	10	11	12	
$a =$	$\overset{b_1}{1}$	$\overset{b_2}{a_2}$	$\overbrace{\underline{1},\ 1}^{b_3}$	$\overset{b_4}{a_5}$	$\overbrace{\underline{1},\ 1}^{b_5}$	$\overset{b_6}{a_8}$	$\overset{b_7}{a_9}$	$\overbrace{\underline{1},\ a_{11}}^{b_8}$	$\overset{b_9}{a_{12}}$	$\overbrace{\underline{1},\ 1}^{b_{10}}$	$\overbrace{\underline{1},\ a_{16}}^{b_{11}}$	$\overset{b_{12}}{a_{17}}$	$\ldots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\ldots$
$(i_a(k))_{k\in\mathbf{N}^+} =$	1	2	3	5	6	8	9	10	12	13	15	17	$\ldots$

$$(i_a(k))_{k\in\mathbf{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots)$$

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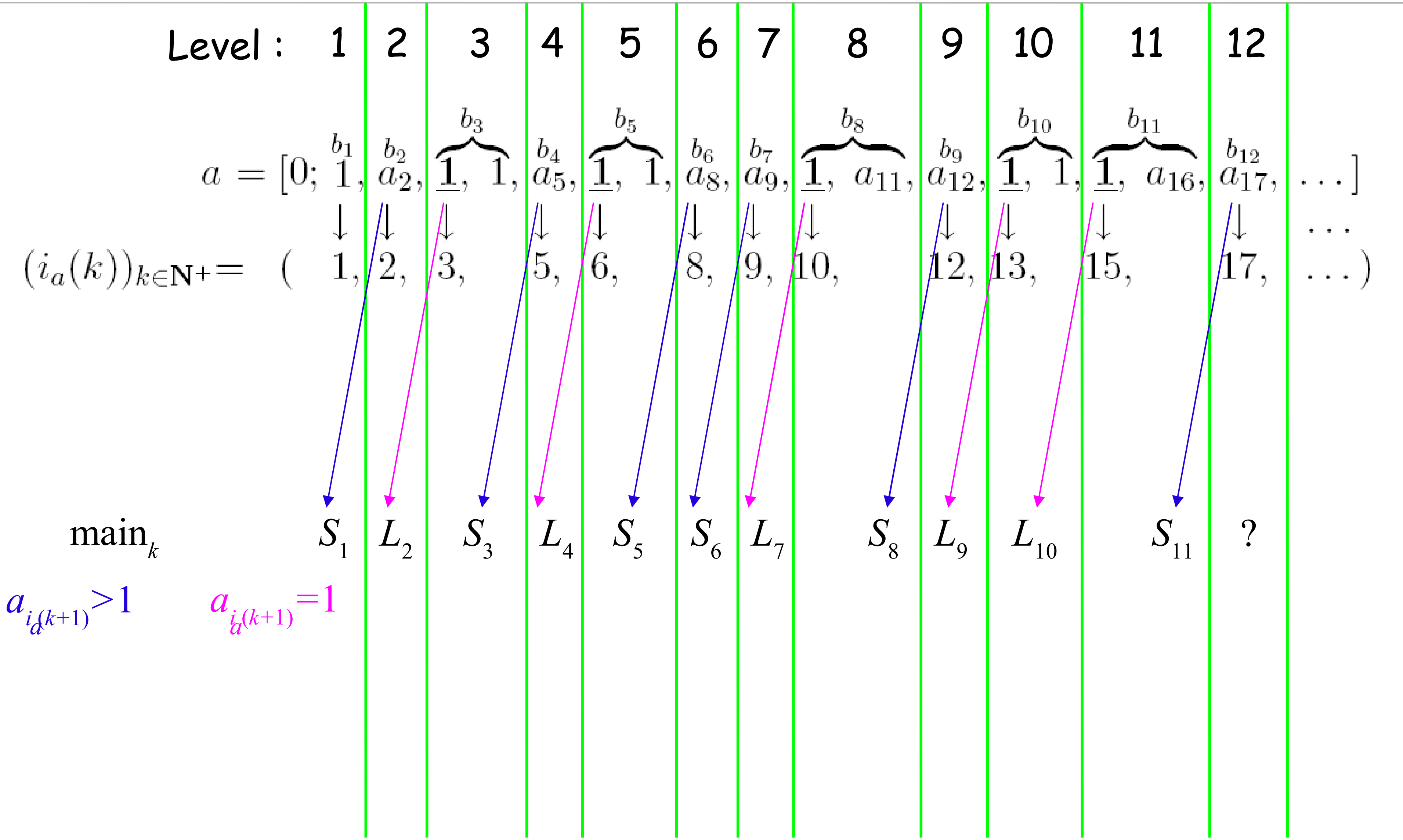
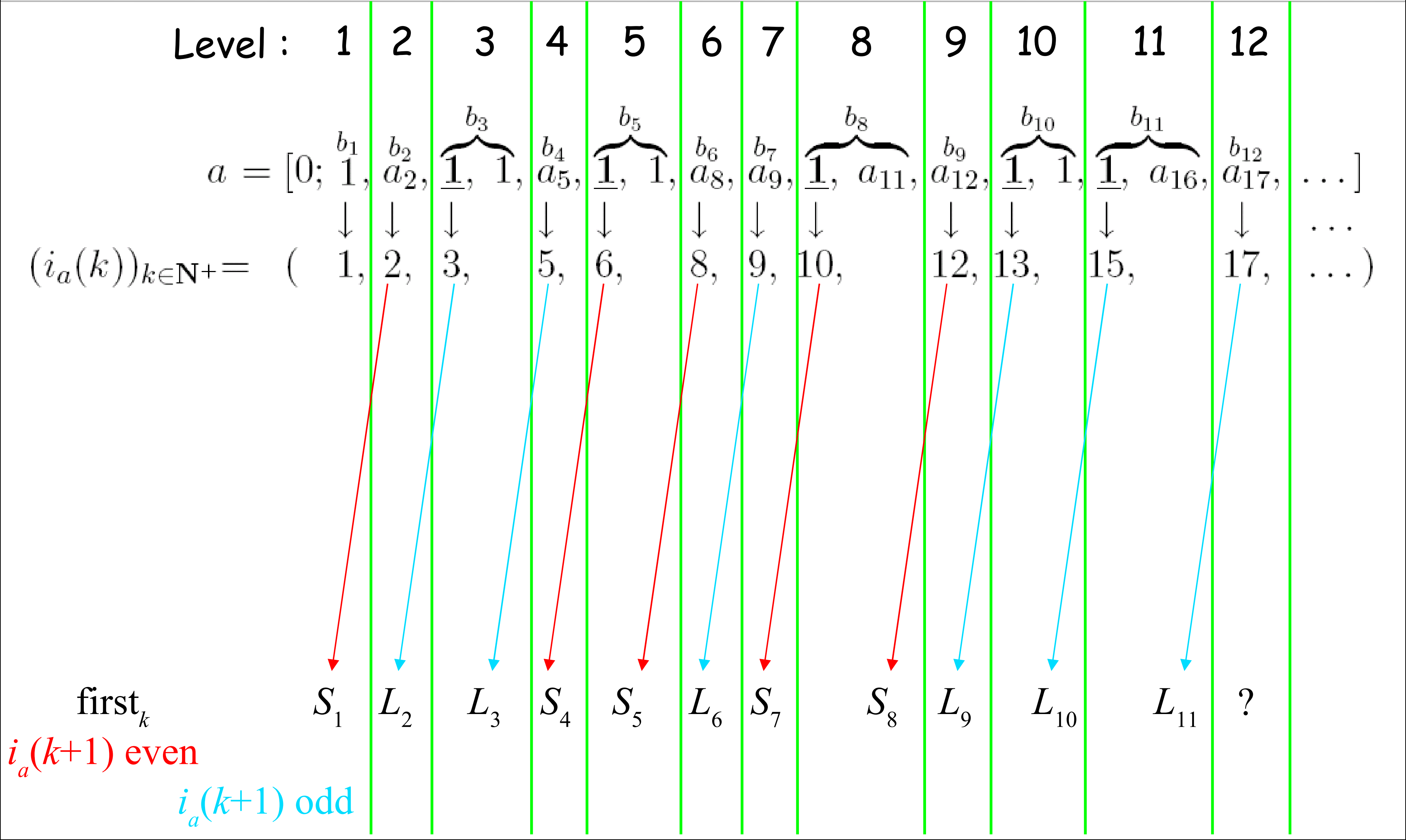
Short run length: the CF elements

Level :	1	2	3	4	5	6	7	8	9	10	11	12	
$a = [0;$	$\overset{b_1}{1},$	$\overset{b_2}{a_2},$	$\overbrace{\underline{1},\ 1}^{b_3},$	$\overset{b_4}{a_5},$	$\overbrace{\underline{1},\ 1}^{b_5},$	$\overset{b_6}{a_8},$	$\overset{b_7}{a_9},$	$\overbrace{\underline{1},\ a_{11}}^{b_8},$	$\overset{b_9}{a_{12}},$	$\overbrace{\underline{1},\ 1}^{b_{10}},$	$\overbrace{\underline{1},\ a_{16}}^{b_{11}},$	$\overset{b_{12}}{a_{17}},$	$\ldots]$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\ldots$
$(i_a(k))_{k\in\mathbf{N}^+} = ($	$1,$	$2,$	$3,$	$5,$	$6,$	$8,$	$9,$	$10,$	$12,$	$13,$	$15,$	$17,$	$\ldots)$
$\lVert S_k \rVert = b_k$	1	a_2	2	a_5	2	a_8	a_9	$1 + a_{11}$	a_{12}	2	$1 + a_{16}$	a_{17}	

The most frequent run: essential 1's



The first run: parity of the function



An illustration: for $a_2=2$ and $a_5=3$:

P_n - the n^{th} prefix according to the run hierarchy

S_n - short run of level n

L_n - long run of level n

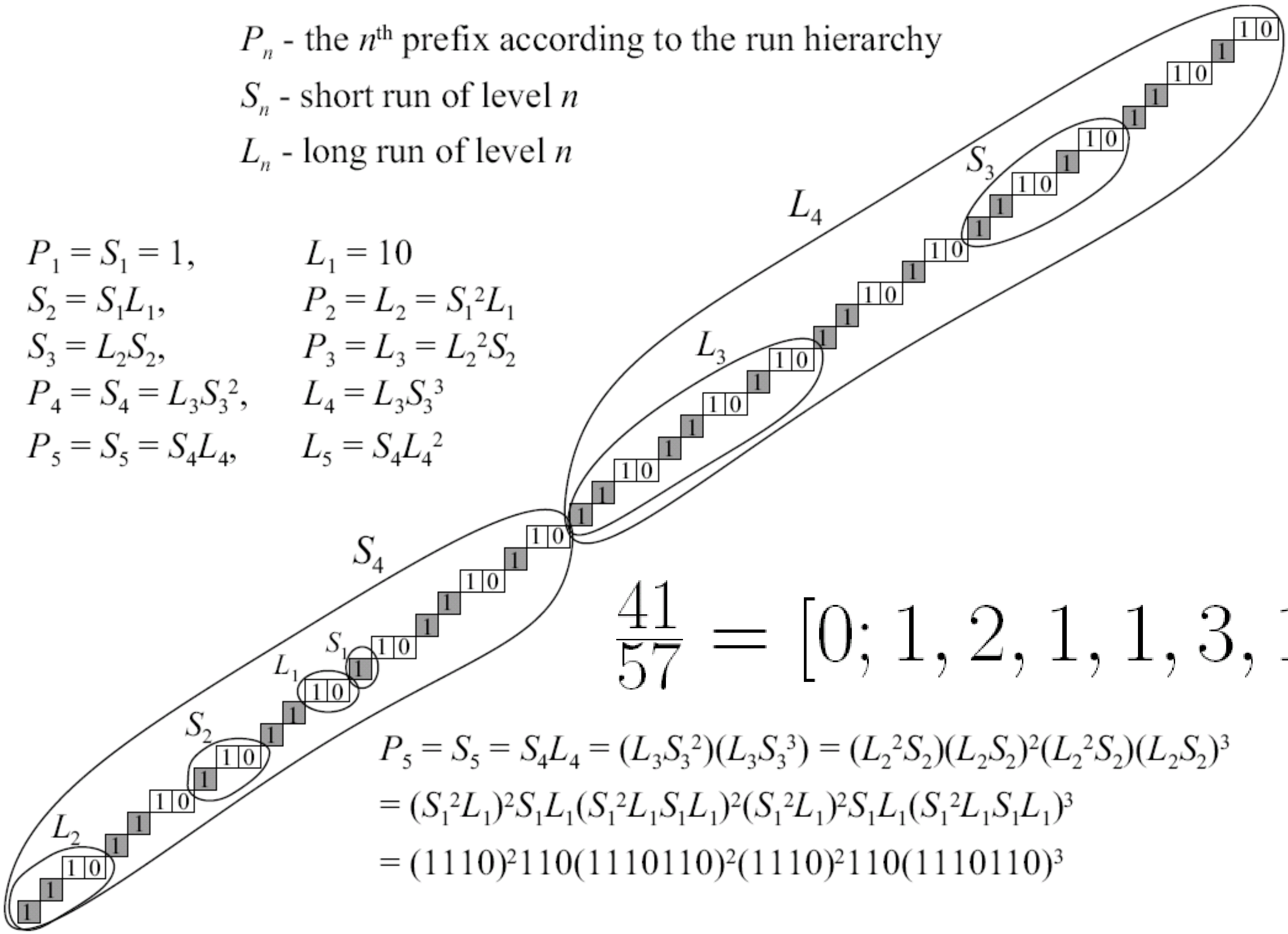
$P_1 = S_1 = 1,$ $\quad L_1 = 10$

$S_2 = S_1L_1,$ $\quad P_2 = L_2 = S_1^2L_1$

$S_3 = L_2S_2,$ $\quad P_3 = L_3 = L_2^2S_2$

$P_4 = S_4 = L_3S_3^2,$ $\quad L_4 = L_3S_3^3$

$P_5 = S_5 = S_4L_4,$ $\quad L_5 = S_4L_4^2$



$$\frac{41}{57} = [0; 1, 2, 1, 1, 3, 1, 1]$$

$$P_5 = S_5 = S_4L_4 = (L_3S_3^2)(L_3S_3^3) = (L_2^2S_2)(L_2S_2)^2(L_2^2S_2)(L_2S_2)^3$$

$$= (S_1^2L_1)^2S_1L_1(S_1^2L_1S_1L_1)^2(S_1^2L_1)^2S_1L_1(S_1^2L_1S_1L_1)^3$$

$$= (1110)^2110(1110110)^2(1110)^2110(1110110)^3$$

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The sequence of length specification for a

$$b_1 = a_1 \quad \text{and, for} \quad n \geq 2:$$

$$b_n = \begin{cases} a_{i_a(n)}, & a_{i_a(n)} \neq 1 \\ 1 + a_{i_a(n)+1}, & a_{i_a(n)} = 1 \end{cases}$$

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The same run length on all digitization levels

Each class is generated by a sequence (b_n) such that:

$$b_1 \in \mathbf{N}^+ \quad \text{and, for} \;\; n \geq 2, \;\; b_n \in \mathbf{N}^+ \setminus \{1\}$$

Each such (b_n) is the sequence of length specification for some slope

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Two equivalence relations on the set of slopes

1. based on run length on all levels for $s'(a)$:

$$a \in [(b_1, b_2, b_3, \dots)]_{\sim_{\text{len}}} \quad \Leftrightarrow$$

$$\forall \; k \in \mathbf{N}^+ \quad \|S_k\| = b_k$$

2. based on run construction on all levels for $s'(a)$:

$$a \sim_{\text{con}} a' \quad \Leftrightarrow \quad i_a \equiv i_{a'}$$

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Done until now and to be done after a break:

1. Background information

2. Intuitions

3. Formal definitions and some motivation

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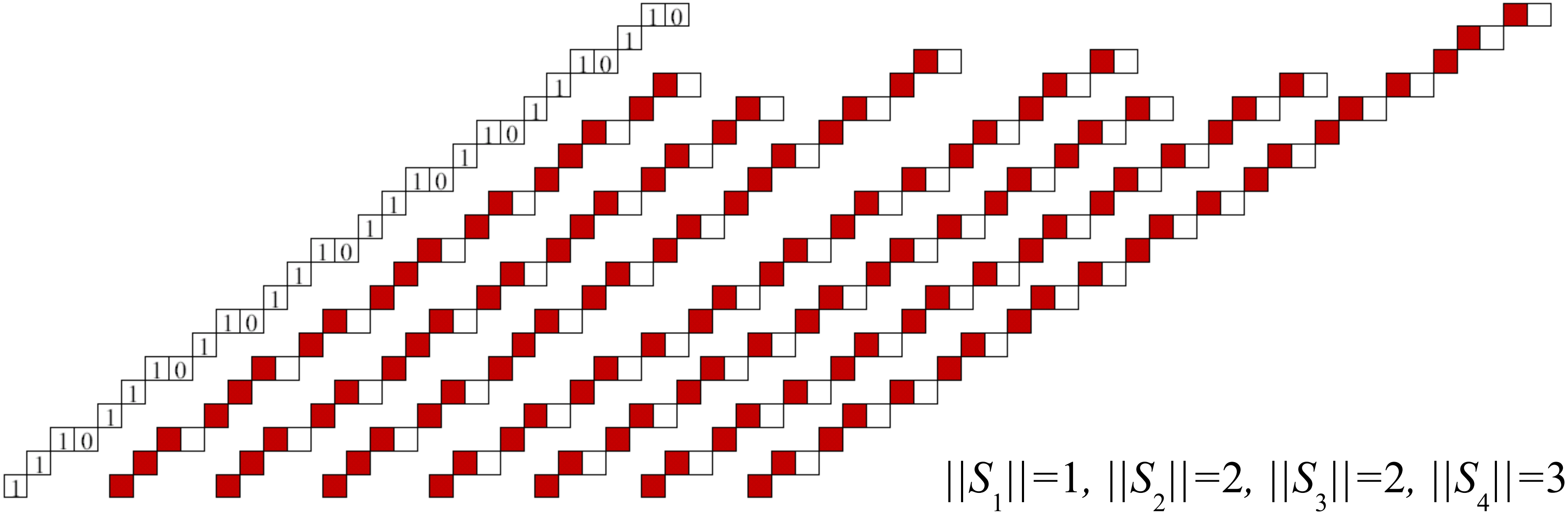
4. Some results and open questions:

- description of classes

- fixed point theorem.

Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)



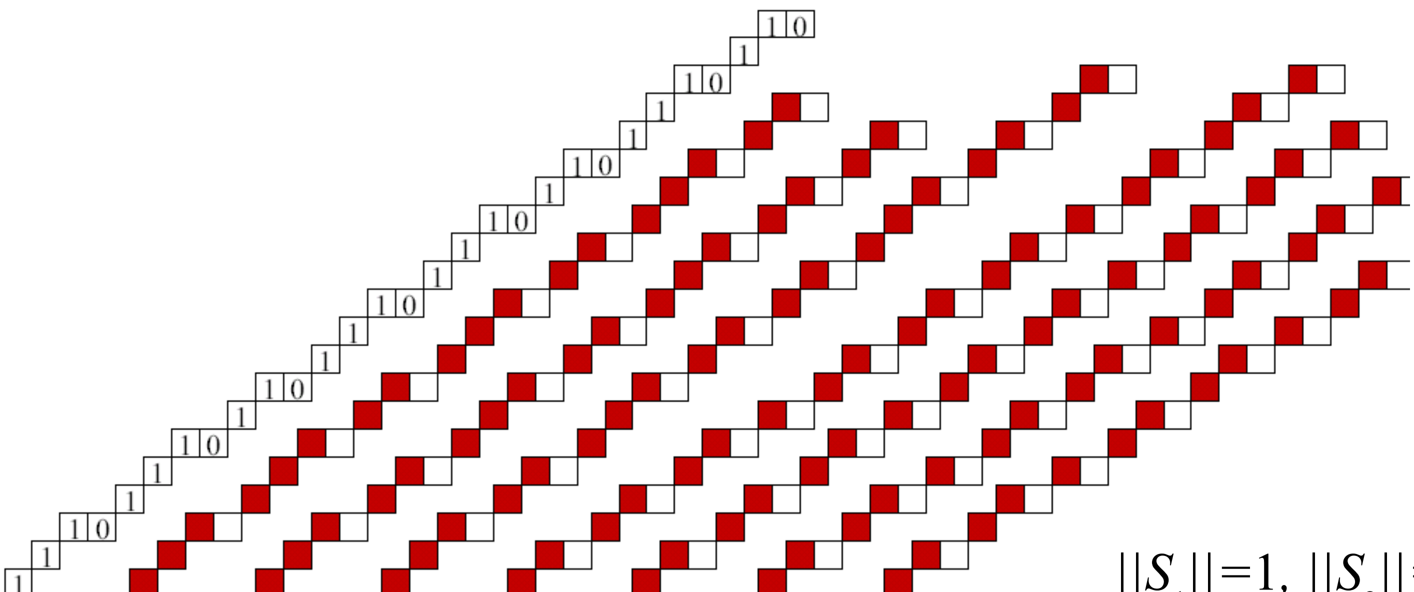
$||S_1||=1,\ ||S_2||=2,\ ||S_3||=2,\ ||S_4||=3.$

$$(b_n)_{n=1}^{\infty} = (1, 2, 2, 3, \ldots)$$

All lines from the same class have the same run lengths on all digitization levels.

Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)



$[0; 1, 2, 1, 1, 1, 2]$

$[0; 1, 2, 1, 1, 3]$

$[0; 1, 2, 2, 3]$

$[0; 1, 2, 2, 1, 2]$

$[0; 1, 1, 1, 2, 1, 2]$

$[0; 1, 1, 1, 2, 3]$

$[0; 1, 1, 1, 1, 1, 3]$

$[0; 1, 1, 1, 1, 1, 1, 2]$

$||S_1||=1,\ ||S_2||=2,\ ||S_3||=2,\ ||S_4||=3.$

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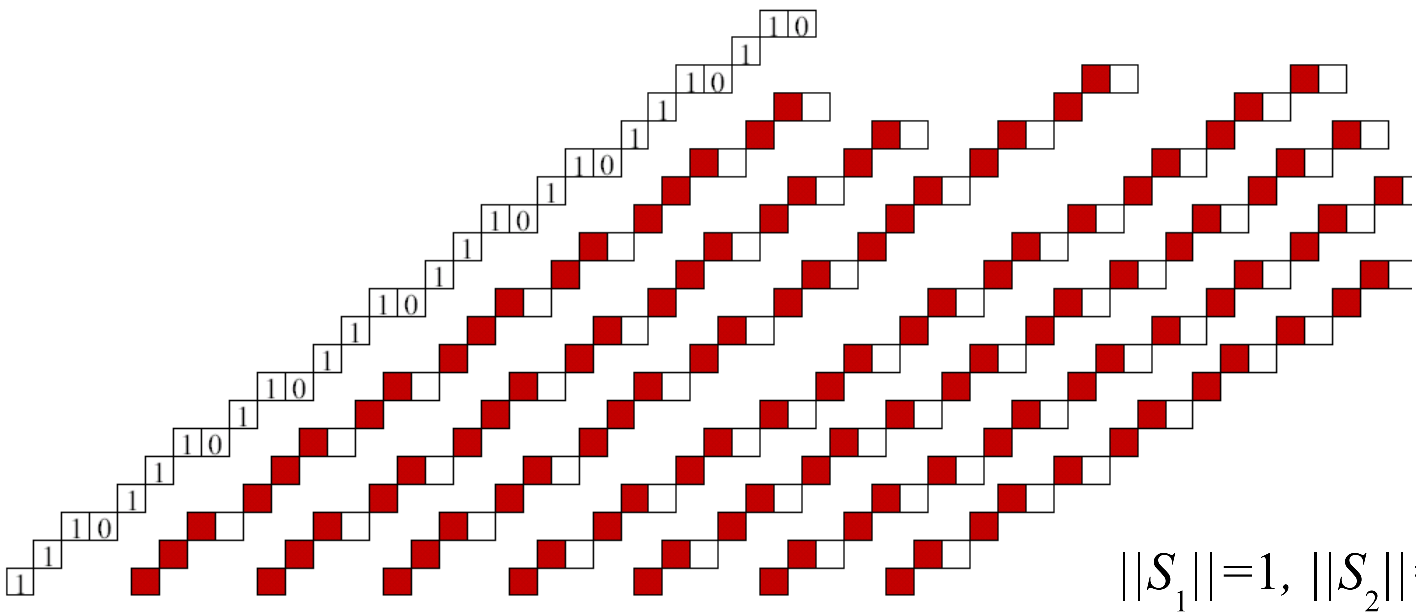


$[0; 1, 2, 1, 1, 1, 2]$

$[0; 1, 2, 1, 1, 3]$

$[0; 1, 2, 2, 3]$

$[0; 1, 2, 2, 1, 2]$

$[0; 1, 1, 1, 2, 1, 2]$

$[0; 1, 1, 1, 2, 3]$

$[0; 1, 1, 1, 1, 1, 3]$

$[0; 1, 1, 1, 1, 1, 1, 2]$

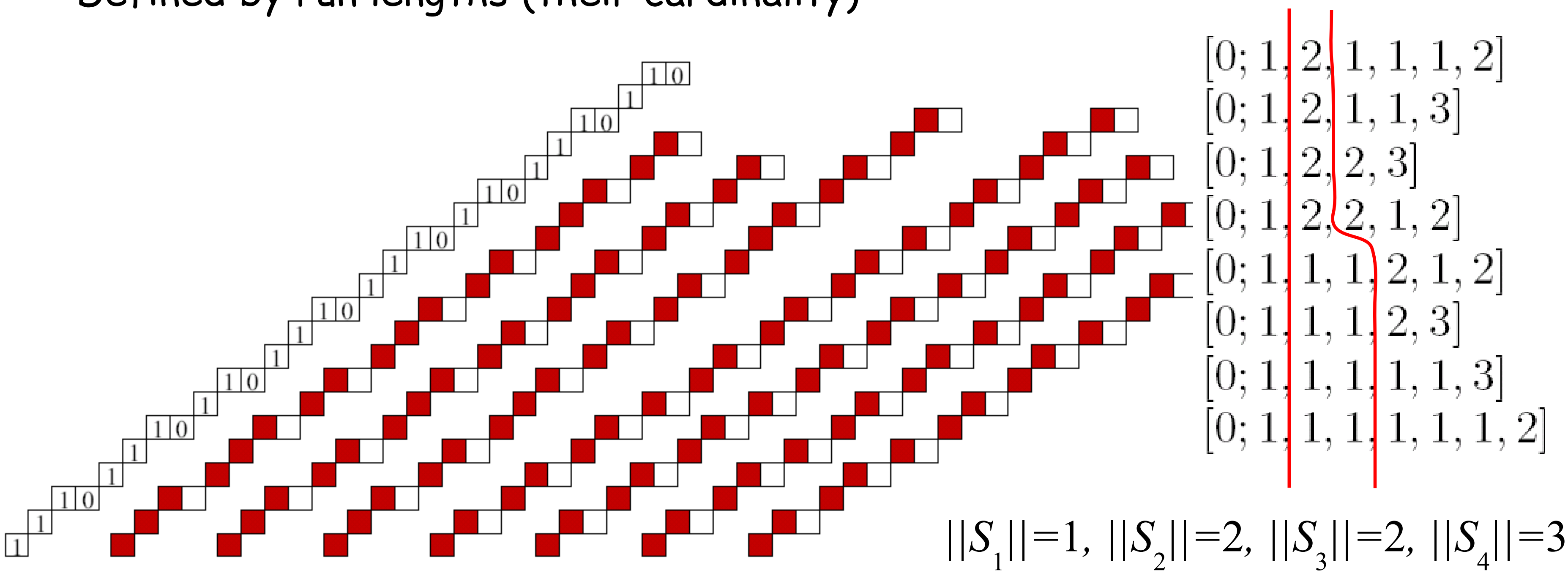
$||S_1||=1, \ ||S_2||=2, \ ||S_3||=2, \ ||S_4||=3.$

$$(b_n)_{n=1}^{\infty} = (1, 2, 2, 3, \dots)$$

All lines from the same class have the same run lengths on all digitization levels.

Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)



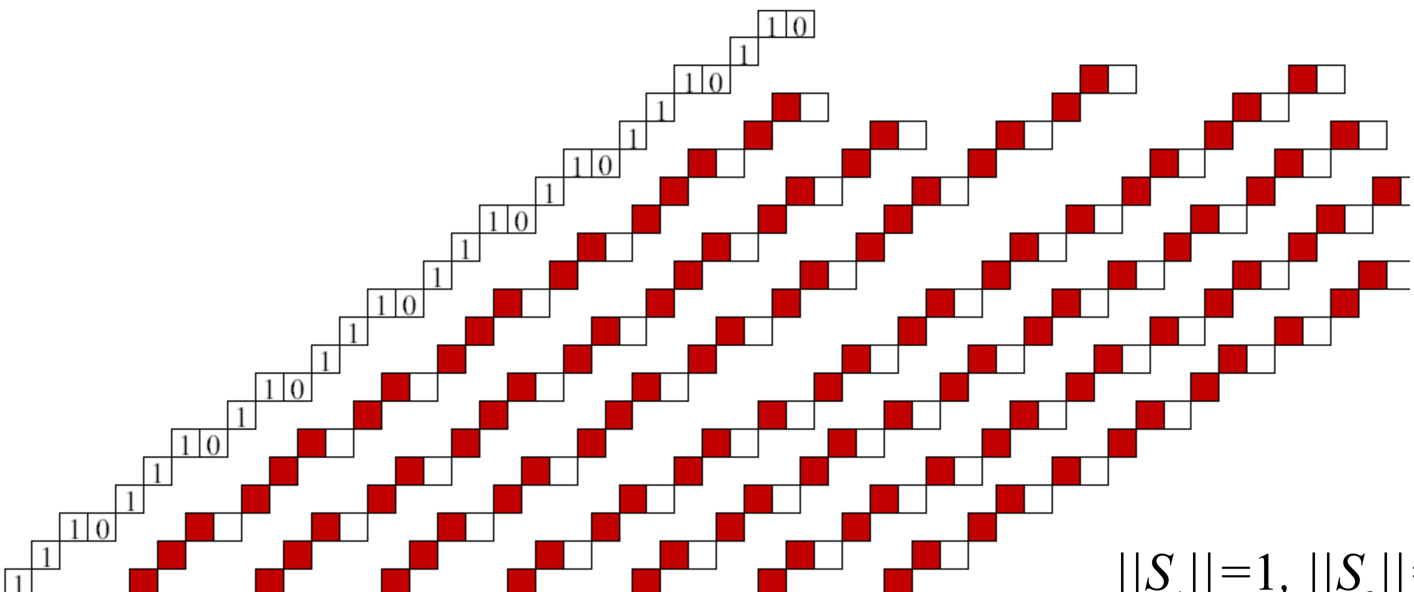
$||S_1||=1,\ ||S_2||=2,\ ||S_3||=2,\ ||S_4||=3.$

$$(b_n)_{n=1}^{\infty} = (1, 2, 2, 3, \ldots)$$

All lines from the same class have the same run lengths on all digitization levels.

Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)



$[0; 1, 2, 1, 1, 1, 2]$

$[0; 1, 2, 1, 1, 3]$

$[0; 1, 2, 2, 3]$

$[0; 1, 2, 2, 1, 2]$

$[0; 1, 1, 1, 2, 1, 2]$

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$[0; 1, 1, 1, 1, 1, 3]$

$[0; 1, 1, 1, 1, 1, 1, 2]$

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All lines from the same class have the same run lengths on all digitization levels.

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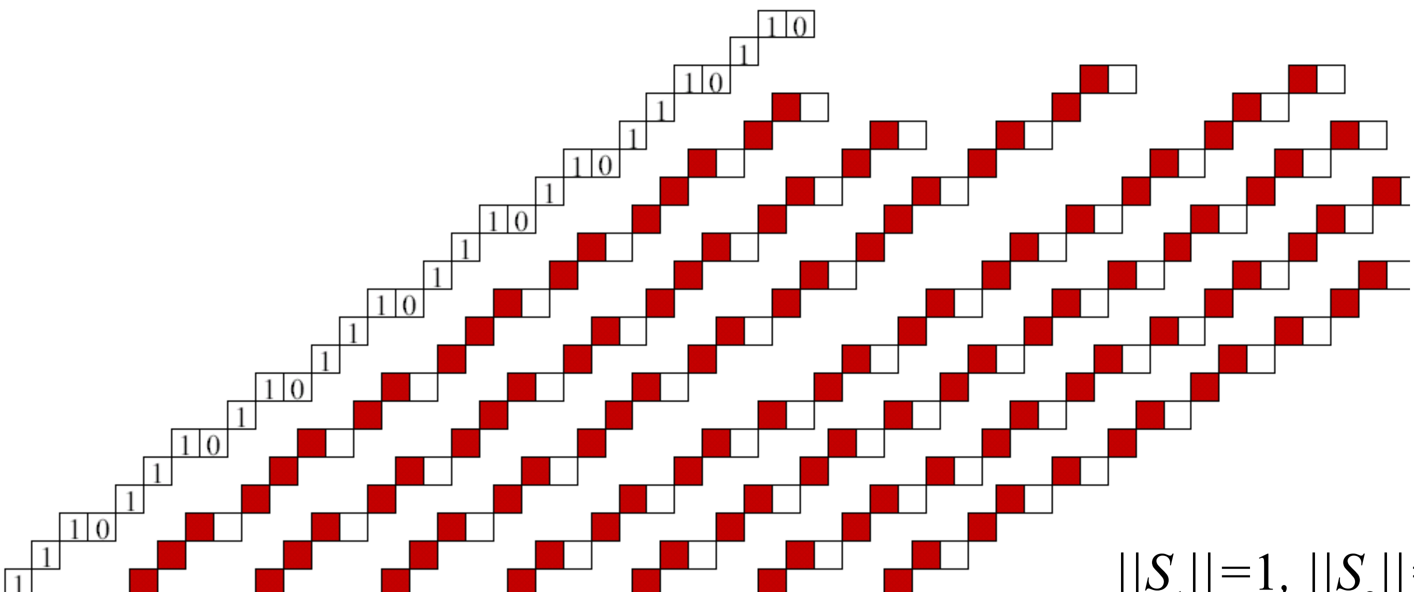
Quantitative equivalence relation (run length)

$2^3 = 8$

Defined by run lengths (their cardinality)



$[0; 1, 2, 1, 1, 1, 2]$

$[0; 1, 2, 1, 1, 3]$

$[0; 1, 2, 2, 3]$

$[0; 1, 2, 2, 1, 2]$

$[0; 1, 1, 1, 2, 1, 2]$

$[0; 1, 1, 1, 2, 3]$

$[0; 1, 1, 1, 1, 1, 3]$

$[0; 1, 1, 1, 1, 1, 1, 2]$

$||S_1||=1, \; ||S_2||=2, \; ||S_3||=2, \; ||S_4||=3.$

$$(b_n)_{n=1}^{\infty} = (1, 2, 2, 3, \ldots)$$

All lines from the same class have the same run lengths on all digitization levels.

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How to compare continued fractions

$$[a_0; a_1, a_2, \ldots] < [b_0; b_1, b_2, \ldots]$$

$$\Updownarrow$$

$$(a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots) \overset{\text{lexic.}}{<} (b_0, -b_1, b_2, -b_3, b_4, -b_5, \ldots)$$

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Quantitative equivalence relation (run length)

The least element of the class :

$$\min\{a \in]0,1[\setminus \mathbf{Q};\ \ a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0; b_1, \overline{1, b_n - 1}]_{n=2}^{\infty}.$$

The largest element of the class :

$$\max\{a \in]0,1[\setminus \mathbf{Q};\ \ a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0; b_1, b_2, \overline{1, b_n - 1}]_{n=3}^{\infty}.$$

Qualitative equivalence relation (run construction)

Defined by the index jump function

Equivalently defined by the places of essential 1's

All lines from the same class have the same construction in terms of long and short runs on all digitization levels.

The least element in each class is 0.

Qualitative equivalence relation (run construction)

A sequence $(t_j)_{j \in J}$ of positive integer numbers will be called an *essential sequence* iff:

- the set J is as follows: $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$,
- the sequence $(t_j)_{j \in J}$ (if not empty) is a sequence of positive integers such that $t_1 \geq 2$ and, for $k \in J \setminus \{1\}$, $t_k - t_{k-1} \geq 2$.

Qualitative equivalence relation (run construction)

Each essential sequence defines an equivalence class under relation con.

An example:

$$\text{If } t_n = 2n - 2 \text{ for each } n \in \mathbf{N}^+, \text{ then}$$

$$[(t_n)_{n=1}^{\infty}]_{\sim_{\text{con}}} = [(\sqrt{5} - 1)/2]_{\sim_{\text{con}}} =$$

$$\{[0; c_1, 1, c_2, 1, c_3, 1, \ldots];\ c_k \in \mathbf{N}^+\}.$$

Qualitative equivalence relation (run construction)

Supremum for each class:

$$\forall \ n \in \mathbf{N}^+ \quad [(\forall \ k \in [1, n-1]_{\mathbf{Z}}, \ \ t_k = 2k)$$

$$\wedge \ (t_n > 2n \ \ \vee \ \ |J| = n-1)]$$

$$\Rightarrow \ \ \sup\{a; \ \ a \in [(t_j)_{j \in J}]_{\sim_{\text{con}}}\} = \frac{F_{2n-1}}{F_{2n}},$$

where $(F_n)_{n \in \mathbf{N}^+}$ is the **Fibonacci** sequence

and $(t_j)_{j \in J}$ is any essential sequence.

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How to compare continued fractions

$$[a_0; a_1, a_2, \ldots] < [b_0; b_1, b_2, \ldots]$$

$$\Updownarrow$$

$$(a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots) \overset{\text{lexic.}}{<} (b_0, -b_1, b_2, -b_3, b_4, -b_5, \ldots)$$

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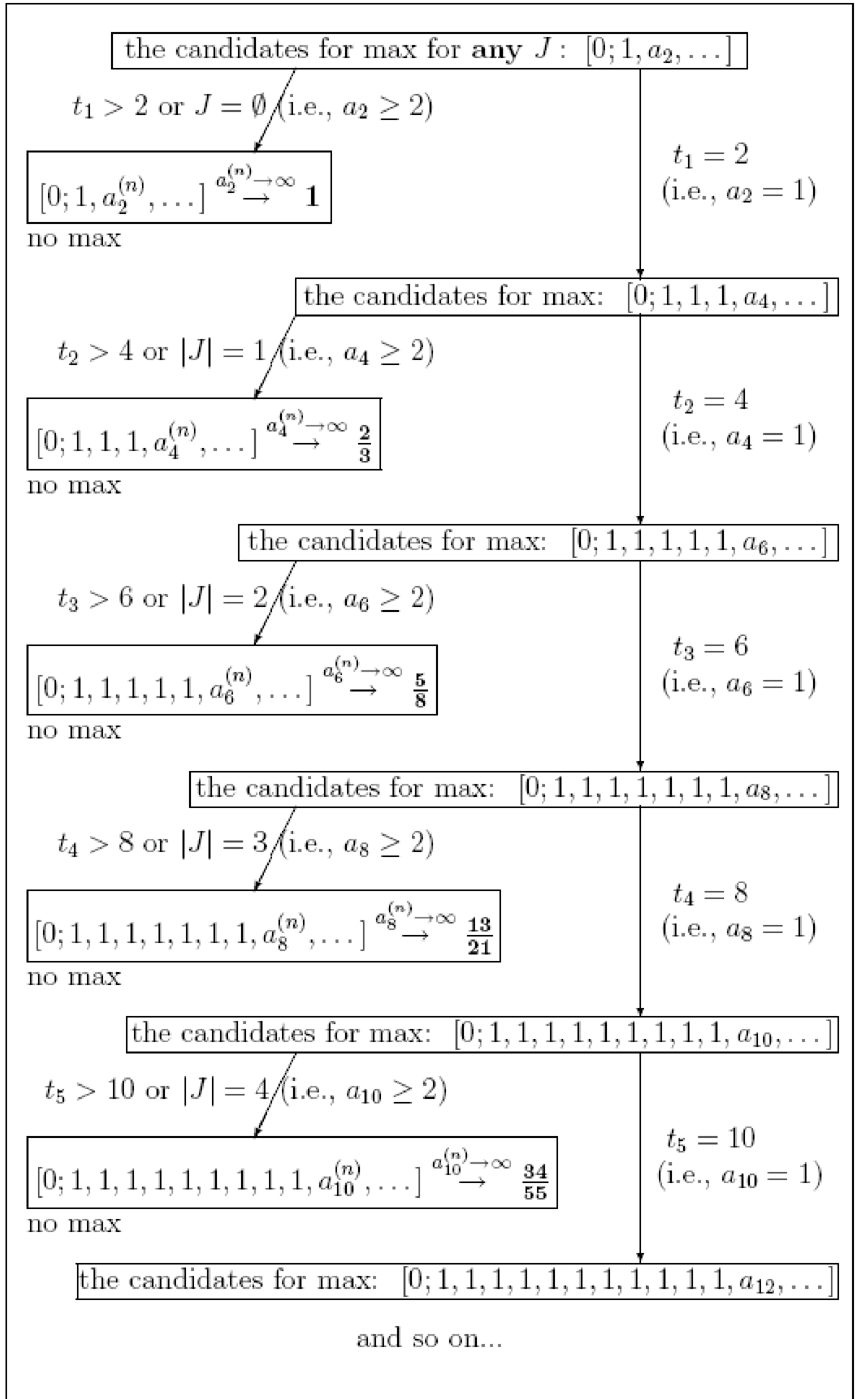

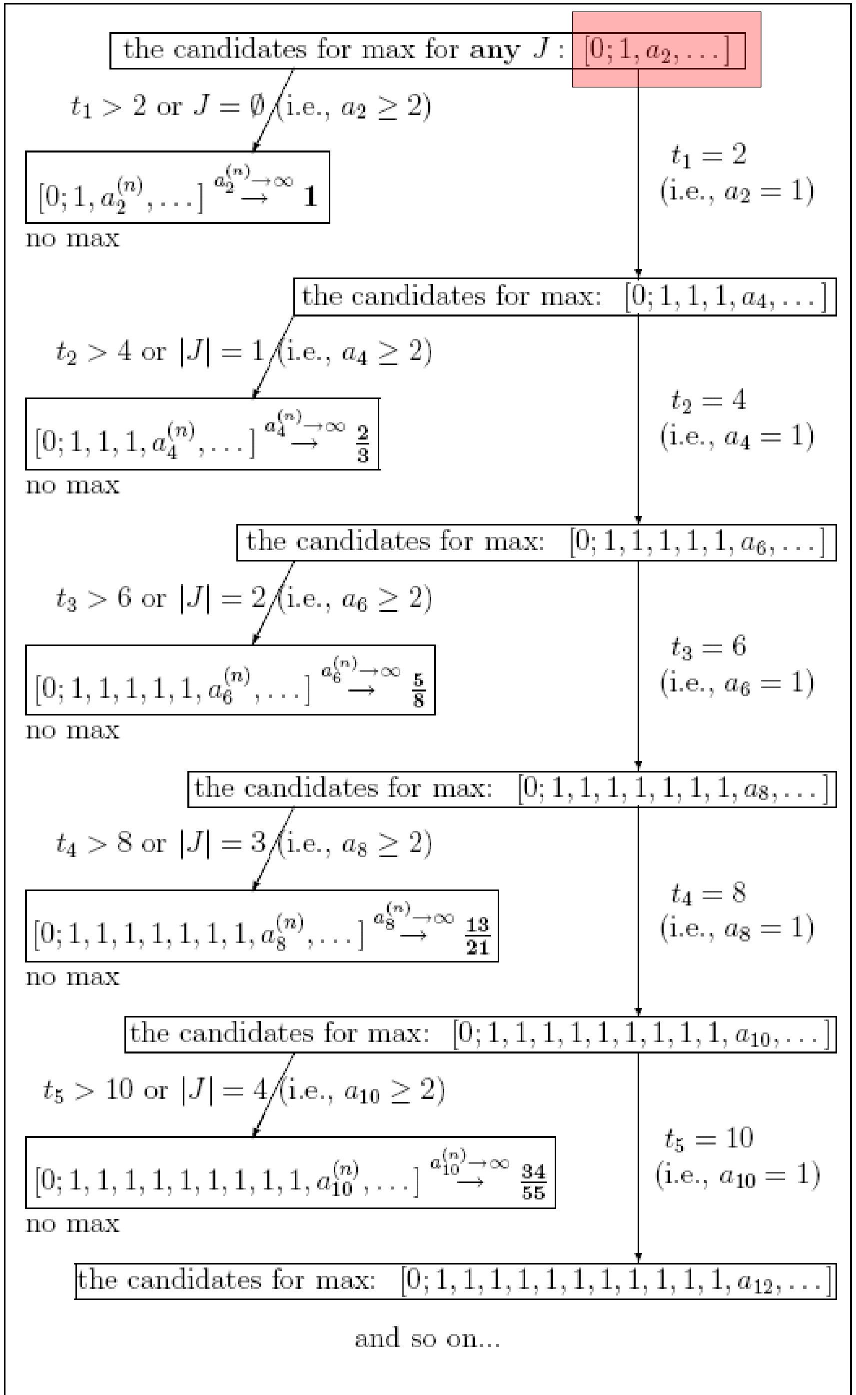
the candidates for max for **any** J : $[0;1,a_2,\dots]$
$t_1>2$ or $J=\emptyset$ (i.e., $a_2\geq 2$)
$[0;1,a_2^{(n)},\dots]\overset{a_2^{(n)}\to\infty}{\longrightarrow}\mathbf{1}$
no max
$t_1=2$
(i.e., $a_2=1$)
the candidates for max: $[0;1,1,1,a_4,\dots]$
$t_2>4$ or $|J|=1$ (i.e., $a_4\geq 2$)
$[0;1,1,1,a_4^{(n)},\dots]\overset{a_4^{(n)}\to\infty}{\longrightarrow}\mathbf{\frac{2}{3}}$
no max
$t_2=4$
(i.e., $a_4=1$)
the candidates for max: $[0;1,1,1,1,1,a_6,\dots]$
$t_3>6$ or $|J|=2$ (i.e., $a_6\geq 2$)
$[0;1,1,1,1,1,a_6^{(n)},\dots]\overset{a_6^{(n)}\to\infty}{\longrightarrow}\mathbf{\frac{5}{8}}$
no max
$t_3=6$
(i.e., $a_6=1$)
the candidates for max: $[0;1,1,1,1,1,1,1,a_8,\dots]$
$t_4>8$ or $|J|=3$ (i.e., $a_8\geq 2$)
$[0;1,1,1,1,1,1,1,a_8^{(n)},\dots]\overset{a_8^{(n)}\to\infty}{\longrightarrow}\mathbf{\frac{13}{21}}$
no max
$t_4=8$
(i.e., $a_8=1$)
the candidates for max: $[0;1,1,1,1,1,1,1,1,1,a_{10},\dots]$
$t_5>10$ or $|J|=4$ (i.e., $a_{10}\geq 2$)
$[0;1,1,1,1,1,1,1,1,1,a_{10}^{(n)},\dots]\overset{a_{10}^{(n)}\to\infty}{\longrightarrow}\mathbf{\frac{34}{55}}$
no max
$t_5=10$
(i.e., $a_{10}=1$)
the candidates for max: $[0;1,1,1,1,1,1,1,1,1,1,1,a_{12},\dots]$
and so on...

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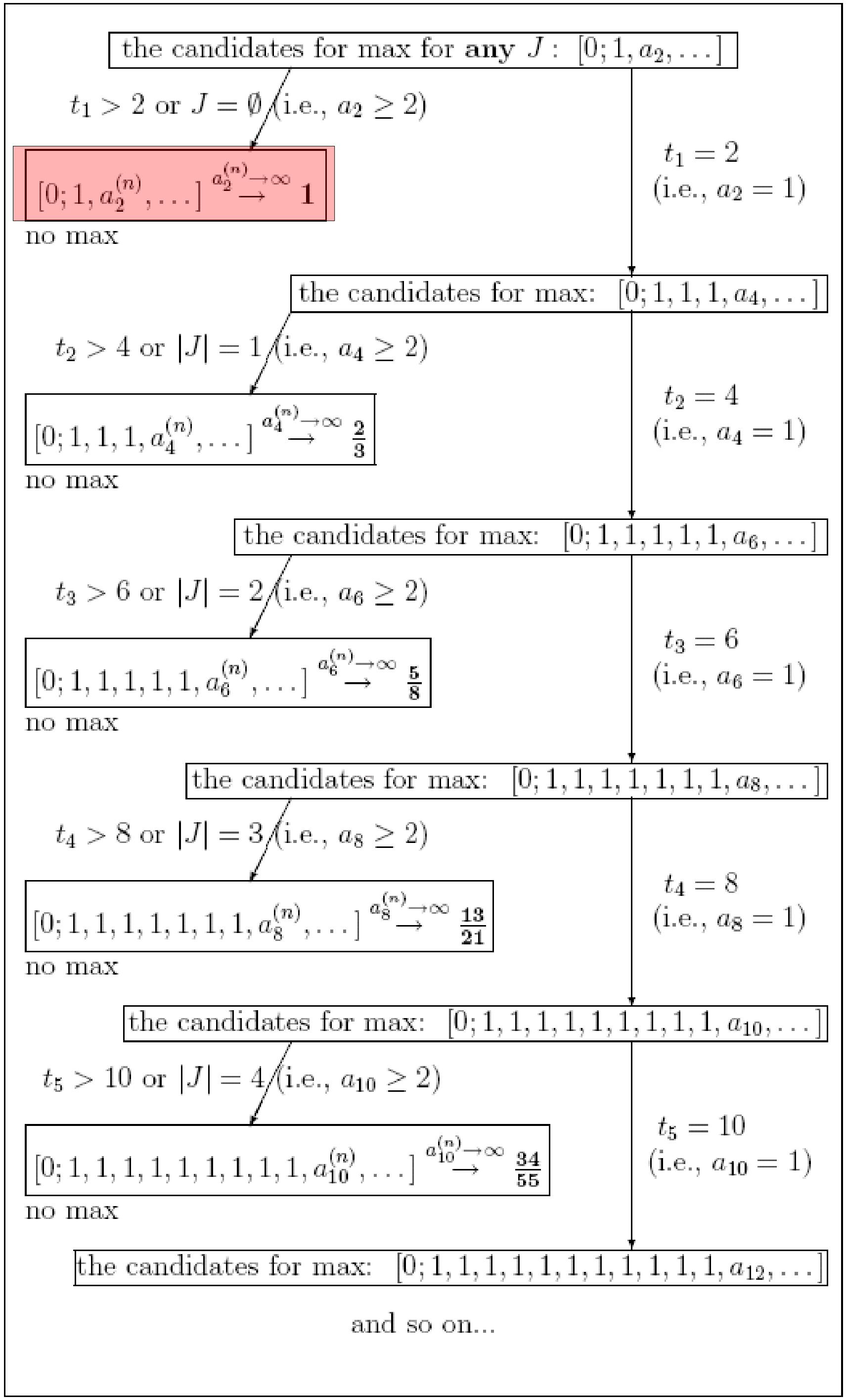
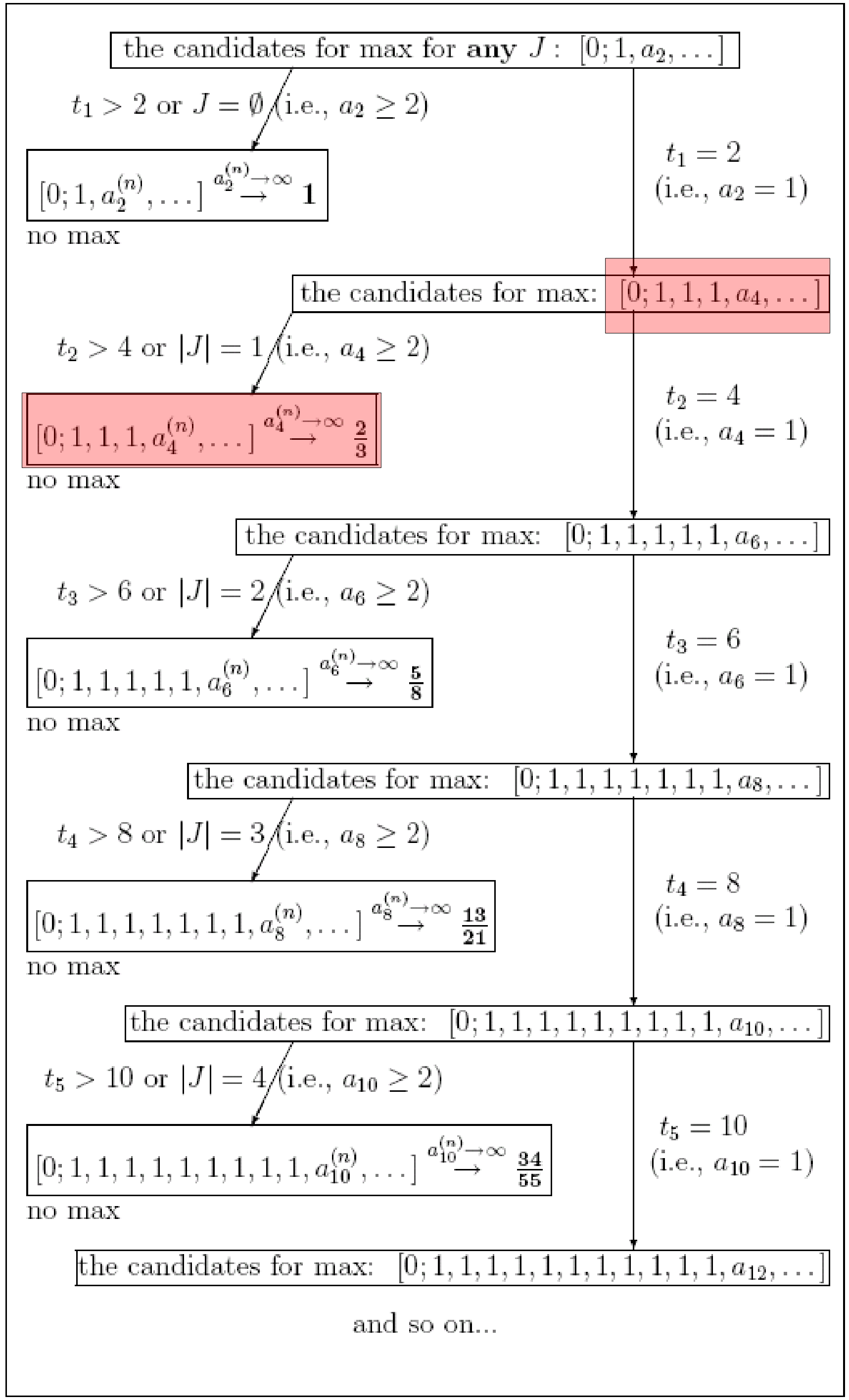
$$\frac{F_1}{F_2}$$



$$\frac{F_1}{F_2}$$

$$\frac{F_3}{F_4}$$



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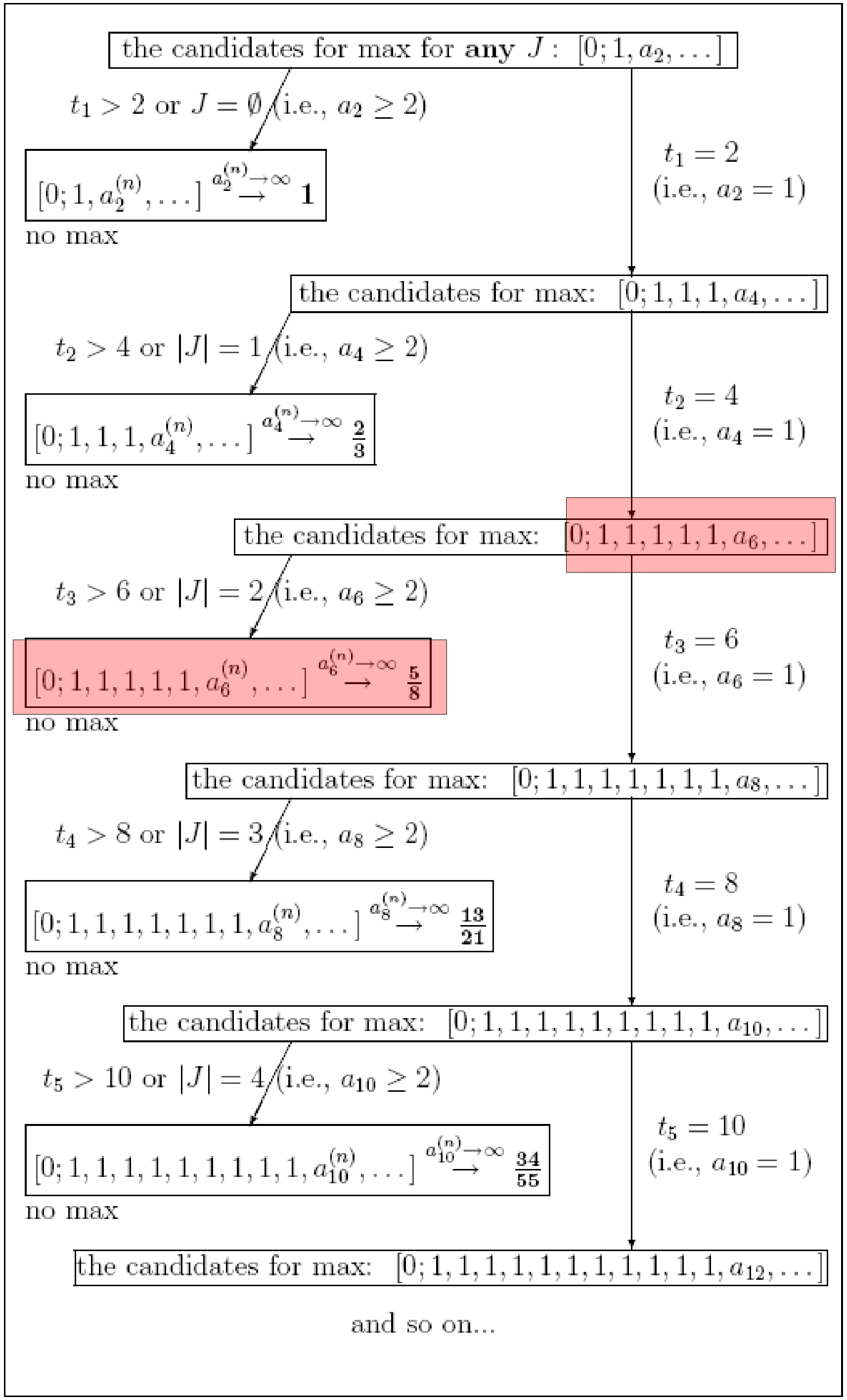
$$\frac{F_1}{F_2}$$

$$\frac{F_3}{F_4}$$

$$\frac{F_5}{F_6}$$



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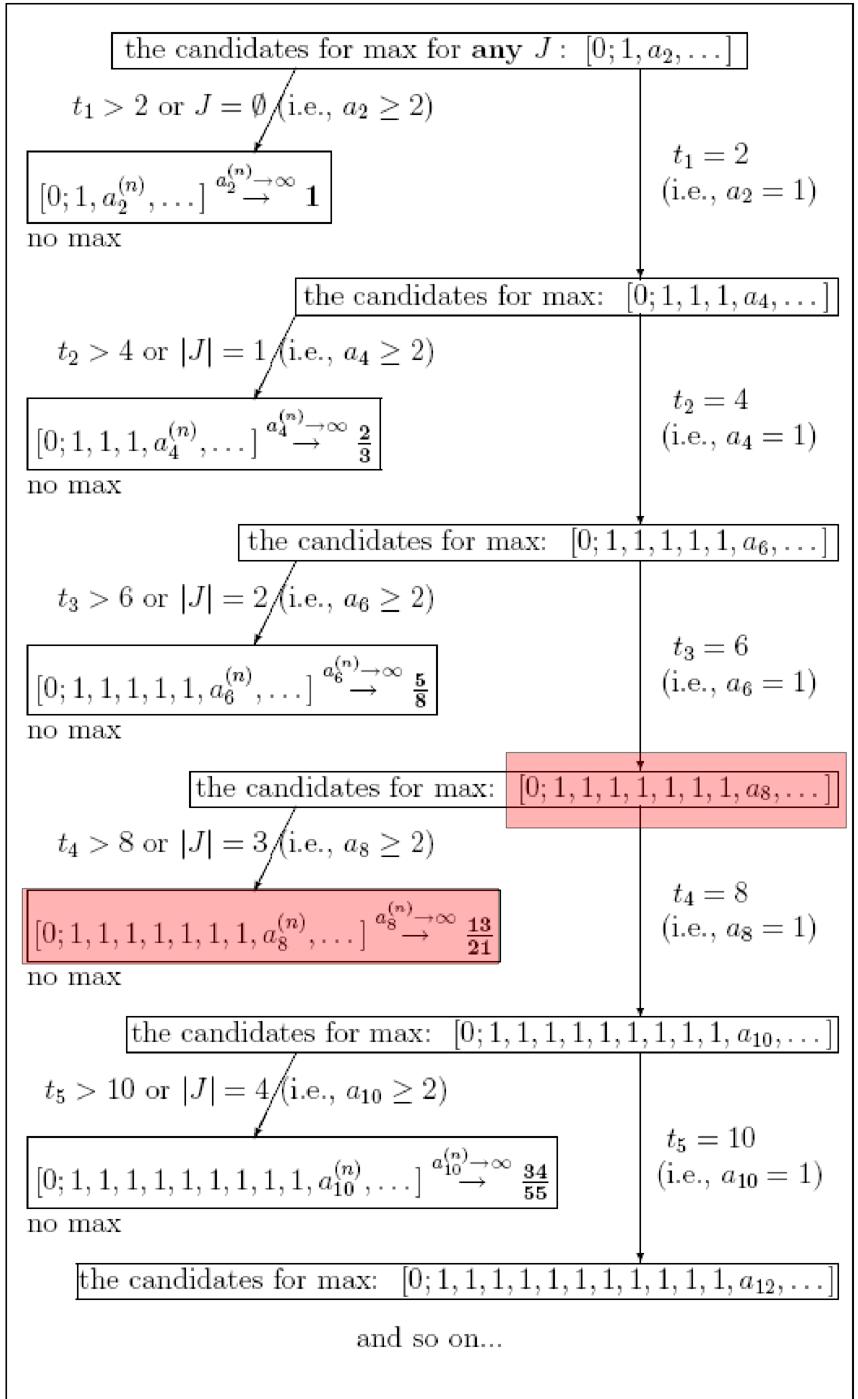
$$\frac{F_1}{F_2}$$

$$\frac{F_3}{F_4}$$

$$\frac{F_5}{F_6}$$

$$\frac{F_7}{F_8}$$



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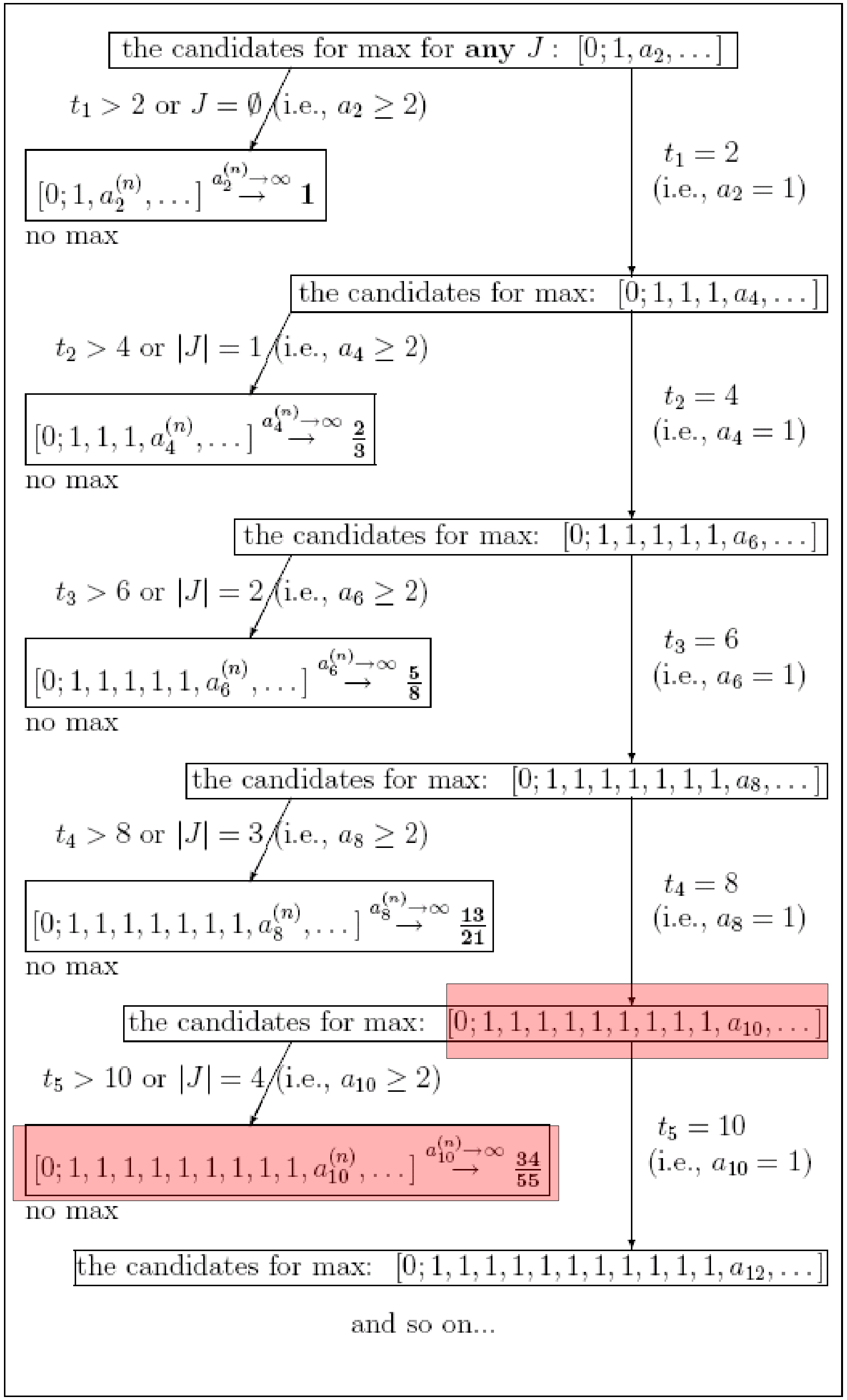
$$\frac{F_1}{F_2}$$

$$\frac{F_3}{F_4}$$

$$\frac{F_5}{F_6}$$

$$\frac{F_7}{F_8}$$

$$\frac{F_9}{F_{10}}$$



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A new fixed point theorem for words

Kolakoski word

The set of all right infinite words over {1,2}:

$$\{1,2\}^\omega$$

$$w\colon \mathbf{N}^+ \to \{1,2\}$$

$$w = w(1)w(2)w(3)\cdots \in \{1,2\}^\omega$$

The run-length encoding operator

$$\Delta_l \colon \{1,2\}^\omega \to \mathbf{N}^\omega$$

$$w = \begin{cases} 1^{k_1}2^{k_2}1^{k_3}2^{k_4}\cdots, & \text{if } w \in 1 \cdot \{1,2\}^\omega \\ 2^{k_1}1^{k_2}2^{k_3}1^{k_4}\cdots, & \text{if } w \in 2 \cdot \{1,2\}^\omega \end{cases}$$

$$\Delta_l(w) = k_1 k_2 k_3 \cdots$$

The run-length encoding operator – an example:

$$w = 11111212222222112221212222222221\ \ldots$$

$$\Delta_l(w) = 5,1,1,7,2,3,1,1,1,9,\ \ldots$$

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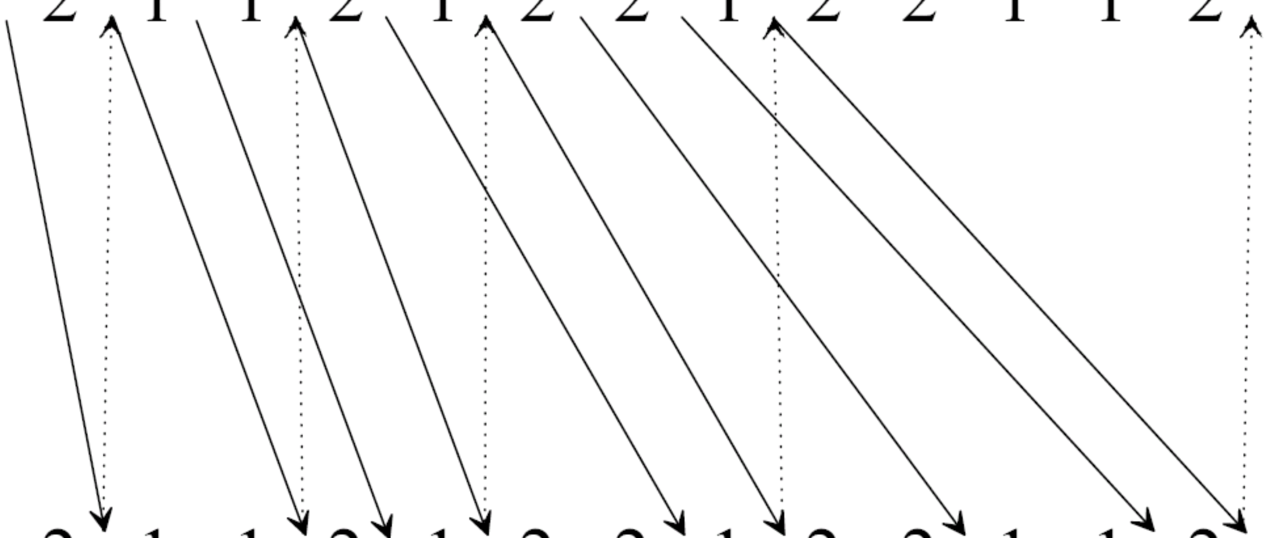
Kolakoski word

$$w = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2 \ldots$$



$$\Delta_l(w) = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2 \ldots$$

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Fixed point theorem: the constructional word

The constructional word $\gamma(a) \in \{0,1\}^\omega$

Let $a = [0; a_1, a_2, \ldots]$. For $n \in \mathbf{N}^+$:

$$\gamma_n(a) = i_a(n+2) - i_a(n+1) - 1$$

$$\gamma_n(a) = \delta_1(a_{i_a(n+1)})$$

$$\gamma_n(a) = \begin{cases} 0, & S_n \text{ is the most frequent} \\ & \text{run on level } n \text{ for } s'(a) \\ 1, & L_n \text{ is the most frequent} \\ & \text{run on level } n \text{ for } s'(a). \end{cases}$$

Fixed point theorem: the constructional word

The constructional word $\gamma(a) \in \{0,1\}^\omega$

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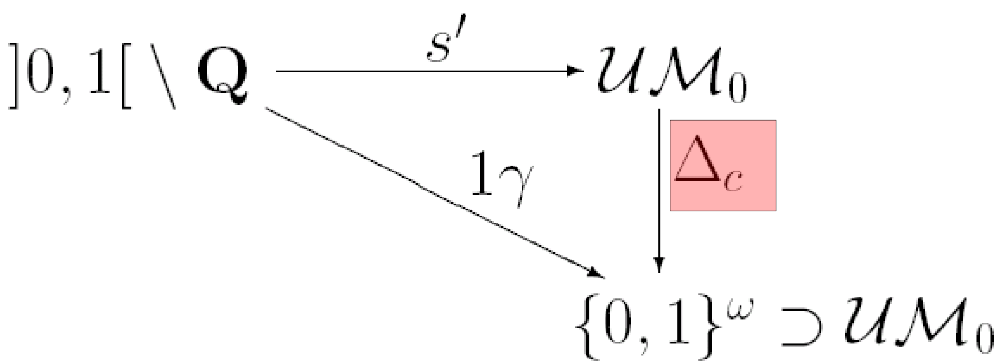
Fixed point theorem: the run-construction encoding operator

Definition The *run-construction encoding operator* $\Delta_c : \mathcal{UM}_0 \longrightarrow \{0,1\}^\omega$ is defined as $\Delta_c = (1\gamma) \circ (s')^{-1}$.



where $\mathcal{UM}_0$ denotes the set of all upper mechanical words with irrational slope $0 < a < 1$ and with intercept 0.

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Balanced construction

Let $a \in \,]0,1[\, \setminus \mathbf{Q}$. The word $s'(a) = 1c(a)$ has

balanced construction if

$$\exists\, \alpha \in \mathbf{R} \quad \gamma(a) = c(\alpha)$$

Sturmian-balanced construction if

$$\exists\, \alpha \in \,]0,1[\, \setminus \mathbf{Q} \quad \gamma(a) = c(\alpha)$$

self-balanced construction

$$1\gamma(a) = \Delta_c(1c(a)) = 1c(a)$$

Balanced construction – some examples

Paper VI. Examples 2, 3, 4, 5.

- The words $s'(a)$ with $a = [0; a_1, a_2, a_3, \ldots]$, where $a_k \geq 2$ for all $k \geq 2$, have balanced construction.
- The words $s'(a)$ with $a = [0; a_1, 1, a_3, 1, a_5, 1, a_7, \ldots]$, where $a_{2k-1} \in \mathbf{N}^+$ for all $k \in \mathbf{N}^+$, have balanced construction.

A fixed-point theorem: exactly 1 fixed point in each equivalence class

Let $(b_n)_{n\in\mathbf{N}^+}$ be such that $b_1 \in \mathbf{N}^+$ and $b_n \in \mathbf{N}^+ \setminus \{1\}$ for all $n \geq 2$. Then

$$\exists^1_{a\in]0,1[\setminus\mathbf{Q}}$$

$$a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}} \;\wedge\; s'(a) = \Delta_c(s'(a)).$$

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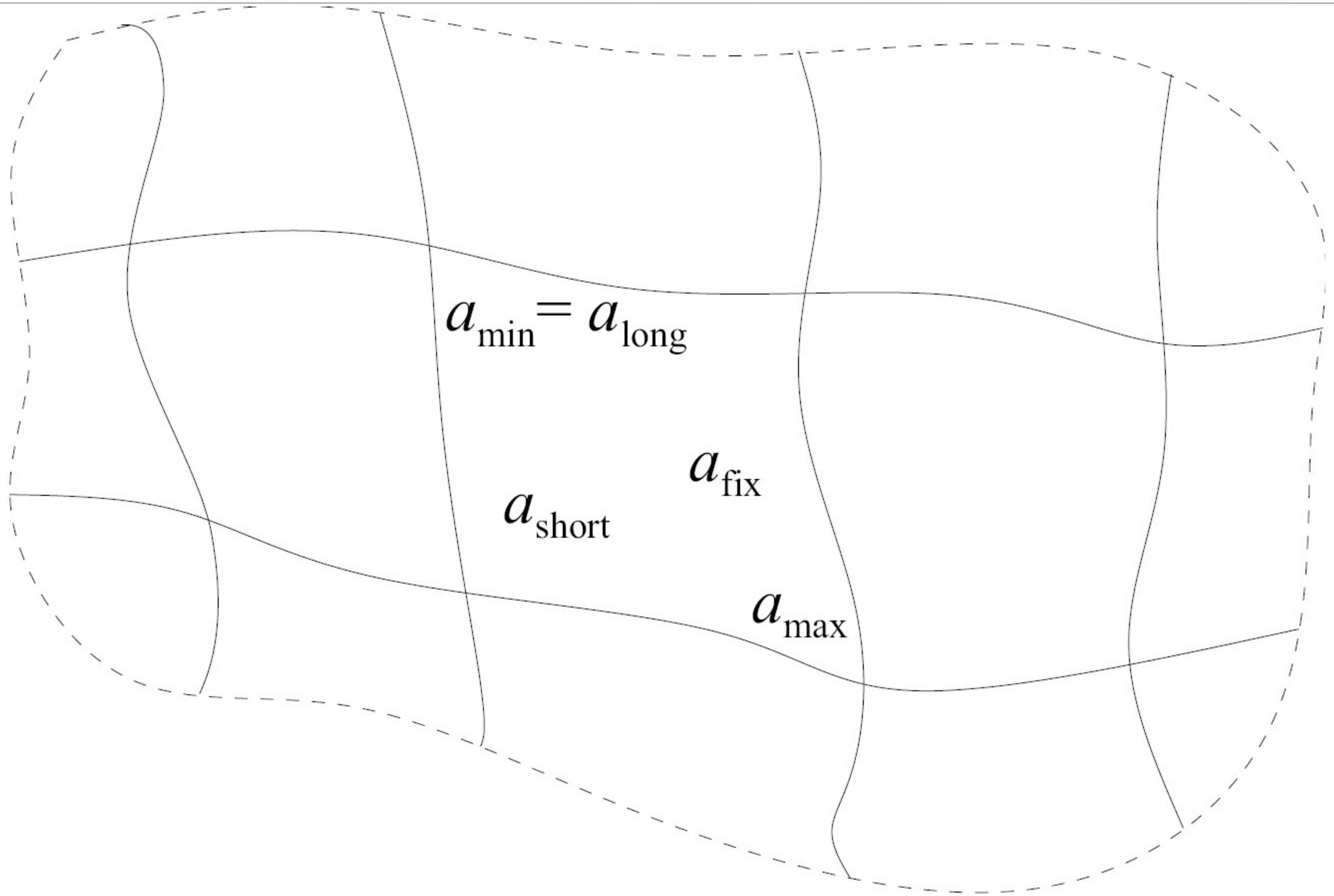
Equivalence classes under the relation len



Equivalence classes under the relation len

- $a_{\max} = [0; b_1, b_2, 1, b_3 - 1, 1, b_4 - 1, 1, b_5 - 1, \ldots]$,
- $a_{\min} = a_{\text{long}} = [0; b_1, 1, b_2 - 1, 1, b_3 - 1, 1, b_4 - 1, \ldots]$,
- $a_{\text{short}} = [0; b_1, b_2, b_3, b_4, \ldots]$,
- a_{fix} is the slope of the fixed point of the run-construction encoding operator Δ_c, i.e., $\gamma(a_{\text{fix}}) = c(a_{\text{fix}})$, where γ is the constructional word.

The set of all fixed points

No quadratic surd can be a fixed point!

Their constructional words have rational slopes, if any.

(Proposition 3 in Paper VI).

The set of all fixed points

Theorem Let $\mathrm{Fix}(\Delta_c) \subset \mathcal{UM}_0$ denote the set of all fixed points of Δ_c. Then:

1. $\mathrm{Fix}(\Delta_c) \subset s'(]0, \frac{2}{3}[\setminus \mathbf{Q})$; numbers 0 and $\frac{2}{3}$ are accumulation points of $(s')^{-1}(\mathrm{Fix}(\Delta_c))$.

2. $\mathrm{card}(\mathrm{Fix}(\Delta_c))$ is equal to that of the continuum.

Some combinatorial questions

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Combinatorics on words – new classes of words

Iterations of the run-construction encoding operator

What can one say about the fixed points?
Formulate an iff condition for CFs of fixed points.

Two kinds of description: by the CF-elements and by the properties of real numbers (transcendental, algebraical)

Thank you for your attention