UPPSALA UNIVERSITET

http://hania.wehlou.com

hania@wehlou.com

Continued Fractions & Digital Lines

Hanna Uscka-Wehlou

Sweden, Uppsala University, Department of Mathematics

We present a new, continued fraction based, recursive description of digital half lines $y = ax,\ x > 0$ for irrational a from the interval $]0,1[$. The description reflects the hierarchy of runs.

The digitization we describe is the following modification of Rosenfeld digitization:

$$D_{R'}(y = ax,\ x > 0) = \{(k, \lceil ak \rceil);\ \ k \in \mathbf{N}^+\}$$

Continued fractions. Positive integers $a_1, a_2, \ldots$ are the continued fraction elements of a, if

$$a = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \overset{\text{def.}}{=} [0; a_1, a_2, a_3, \ldots].$$

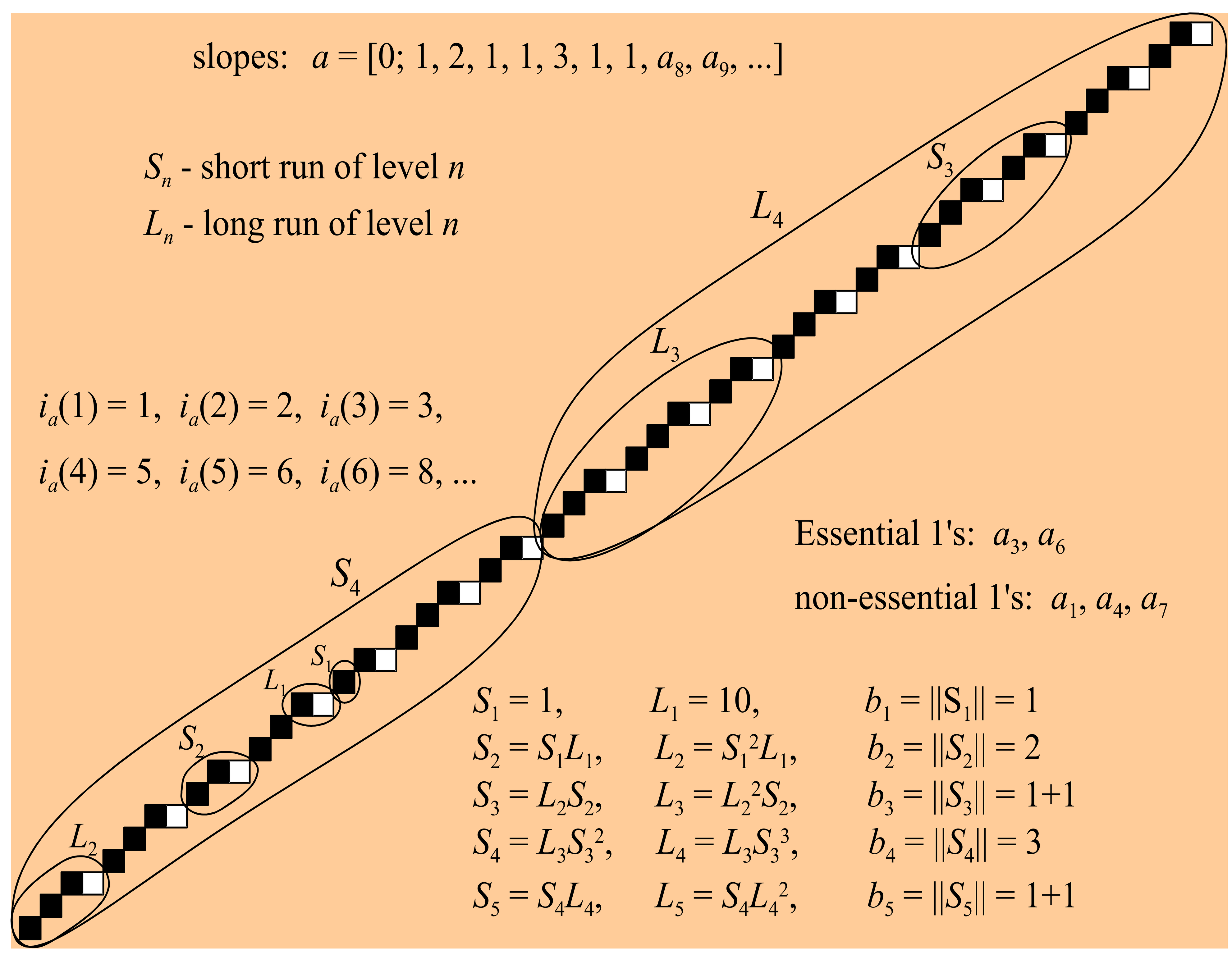
The denominators of the convergents of the continued fraction expansion of a: $q_0 = 1,\ q_1 = a_1,$ and, for $n \geq 2$, $q_n = a_n q_{n-1} + q_{n-2}$

The index jump function

For the slope $a = [0; a_1, a_2, \ldots]$, the index jump function is

$$i_a : \mathbf{N}^+ \to \mathbf{N}^+ \quad \text{such that} \quad i_a(1) = 1,\ \ i_a(2) = 2,$$
and, for $n \geq 2$, $i_a(n+1) = i_a(n) + 1 + \delta_1(a_{i_a(n)})$,

$$\text{where}\ \ \delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1\ . \end{cases}$$



An example how it works for the lines like on the picture. The last row shows the first 12 elements of the sequence of the values of the index jump function.

Let $a_k \geq 2$ for $k = 2, 5, 8, 9, 11, 12, 16, 17.$

$$a = [0; \overset{b_1}{1}, \overset{b_2}{a_2}, \overbrace{1, 1}^{b_3}, \overset{b_4}{a_5}, \overbrace{1, 1}^{b_5}, \overset{b_6}{a_8}, \overset{b_7}{a_9}, \overbrace{1, a_{11}}^{b_8}, \overset{b_9}{a_{12}}, \overbrace{1, 1}^{b_{10}}, \overbrace{1, a_{16}}^{b_{11}}, \overset{b_{12}}{a_{17}}, \ldots]$$
$$\downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \ldots$$
$$(1, 2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17, \ldots)$$

Run hierarchic description in terms of long and short runs on all the digitization levels. Expressed by the index jump function. Illustrated on the picture.

Sequences (b_n) of **length specification** define a **quantitative equivalence relation** LENGTH.

The places of **essential** 1's define a **qualitative equivalence relation** CONSTRUCTION. FIBONACCI numbers and **THE GOLDEN SECTION**

Two kinds of run length

1. $|S_n|, |L_n|$ - the number of pixels contained in S_n, L_n respectively.

2. $\|S_n\|$ - the number of runs$_{n-1}$ contained in S_n. $\|L_n\| = \|S_n\| + 1$.

The pixel-wise length of runs

Expressed by means of the denominators of the convergents and the index jump function. The number of pixels in the line segment $y = ax,\ x > 0$, achieved in the n^{th} step:

$$|S_n| = q_{i_a(n+1)-1}, \quad |L_n| = q_{i_a(n+1)-1} + q_{i_a(n+1)-2}.$$

The cardinality of runs

Calculated according to the following formula:

$$\|S_n\| = b_n = a_{i_a(n)} + \delta_1(a_{i_a(n)}) \cdot a_{i_a(n)+1}.$$

This poster is based on my 3 articles (pre-prints available):

1. Continued Fractions and Digital Lines with Irrational Slopes (the DGCI 2008 paper)
2. A New Description of Upper Mechanical Words with Irrational Slopes Using Continued Fractions (submitted in March 2008)
3. Two Equivalence Relations on Digital Lines with Irrational Slopes. A Continued Fraction Approach to Upper Mechanical Words (March 2008)